Math 300 Winter 2011 Exam 2

Name: ______________________________

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- USE YOUR TIME WISELY! START EACH PROBLEM BY OUTLINING THE PROOFS/IDEAS ON THE PAGE. SPEND NO MORE THAN 10 MINUTES ON EACH PAGE.
- Put your answers on the exam, there are 4 pages (you may use your own paper for scratch work). If you need more space for your answer use the back of the preceding page and indicate that you have done so.
- If you don't know how to prove something, then you should still let me know everything you know about the problem (that is, give definitions and facts you know about the problem and discuss different techniques to prove the theorem). Also, if you are stuck you should try some examples. Don't leave any question blank, give me the opportunity to give you credit for what you know!

GOOD LUCK!

1. Find a specific counterexample to each of the following statements:

   (a) (5 pts) Let $f(x) = 2x$. If A and B are subsets of $\mathbb{R}$ such that $f : A \to B$, then f is a bijection. (Hint: You'll give sets A and B).

   (b) (5 pts) Every strictly monotone function from $\mathbb{R}$ to $\mathbb{R}$ is unbounded. (If you can't come up with a specific function, draw the graph of a function for partial credit)

   (c) (5 pts) $\forall a, b, c \in \mathbb{N}$, if there exist $x, y \in \mathbb{Z}$ such that $ax + by = c$, then $\gcd(a, b) = c$.

2. (6 pts) Determine the coefficient of x^9 in the expansion of $(x - 1)^{12}$.

3. (12 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. Suppose there exists a constant, a, such that the following relationship holds:

$$|f(x) - f(y)| \geq |g(x + a) - g(y + a)| \text{ for all } x, y \in \mathbb{R}.$$

Prove that if g is injective, then f is injective.

4. (10 pts) Let a, b, and c be nonzero integers. Prove that if $a \mid b$ and $d = \gcd(b, c)$, then $ad \mid bc$

5. Assume $f : (0, \infty) \to \mathbb{R}$ is a function that satisfies the relationship

$$f\left(\frac{a}{b}\right) = f(a) - f(b) \text{ for all } a, b \in (0, \infty).$$

(Hint: For each part below you will be making specific substitutions in for a and b. You are **given** the fact that anything from $(0, \infty)$ can be substituted in for a and b).

(a) (5 pts) Prove that $f(1) = 0$.

(b) (12 pts) Prove that if $f(t) > 0$ for all real numbers $t > 1$, then f is an increasing function on the set $(0, \infty)$.

REMEMBER TO VERY CLEARLY GIVE THE STRUCTURE, ORDER, AND SPECIFIC JUSTIFICATIONS OF STEPS IN YOUR PROOF.

6. (20 pts) For sets A, B, and C, let $f : A \to B$ and $g : B \to C$ be functions.
   Define $h = g \circ f : A \to C$. That is, $h(a) = g(f(a))$ for all $a \in A$.

   Prove that if f is a bijection and h is a bijection, then g is a bijection.