Practice Example Solutions

III (a) Converse : The reverse implication.

$P \Rightarrow Q$ ← Implication
$Q \Rightarrow P$ ← Converse

NO RELATIONSHIP BETWEEN TRUTH VALUES

Contrapositive : The negation of the conclusion implies the negation of the hypothesis. (This is equivalent)

$P \Rightarrow Q$ ← Implication
$\neg Q \Rightarrow \neg P$ ← Contrapositive

SAME TRUTH VALUE

Contradiction : Assume P and $\neg Q$ and arrive at a contradiction.

$$\underbrace{P \text{ and } \neg Q \rightarrow \text{FALSE}}_{\text{means that}} \quad P \Rightarrow Q \text{ is TRUE.}$$

Tautology : Two statements that have identical truth values give a tautology.

(b) Recall the definitions

	A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
(1)	F	F	F	F	T	T
(2)	F	T	F	T	T	F
(3)	T	F	F	T	F	F
(4)	T	T	T	T	T	T

So

(i) $\boxed{R \Rightarrow P}$ is defined to be $\boxed{\text{True}}$

F T ← Row (2) above

(ii) $\boxed{(P \text{ or } R) \text{ and } S}$

Row(3) $\underbrace{(T \quad F)}$ F is defined to be $\boxed{\text{False}}$

Row(3) (T and F)

(iii) $\boxed{Q \Rightarrow (P \Rightarrow \neg S)}$

T $\underbrace{(T \Rightarrow T)}$ Row (4) is defined to be $\boxed{\text{True}}$.

$\underbrace{T \Rightarrow T}$

(iv) $\neg(R \text{ or } Q) \Leftrightarrow S$ is defined to be $\boxed{\text{True}}$.

F T
T
F F

(c)

P	Q	$\neg Q$	$\neg P$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$
F	F	T	T	T	T
F	T	F	T	F	F
T	F	T	F	T	T
T	T	F	F	T	T

identical

2 (a) Thm $(A \cup B) - C \subseteq (A - C) \cup B$

pf Let $x \in (A \cup B) - C$.

Then $x \in A \cup B$ and $x \notin C$.

So ($x \in A$ OR $x \in B$) AND $x \in C^c$

If $x \in A$ AND $x \in C^c$, then, by def'n,

$x \in A - C$ so ($x \in A - C$ OR $x \in B$).

If $x \in B$ AND $x \in C^c$, then

$x \in B$ so ($x \in A - C$ OR $x \in B$).

Hence, in all cases,

$$x \in (A - C) \cup B$$

Thus, $(A \cup B) - C \subseteq (A - C) \cup B$ //

Example $A = \{1, 2\}$ $B = \{2, 3\}$

$C = \{2\}$

$(A \cup B) - C = \{1, 2, 3\} - \{2\} = \{1, 3\}$ NOT

$(A - C) \cup B = \{1\} \cup \{2, 3\} = \{1, 2, 3\}$ EQUAL.

(b) Thm $A \cup (A \cap B) = A$

pf "$\subseteq$" Let $x \in A \cup (A \cap B)$. So $x \in A$ or $x \in A \cap B$.

Thus, $x \in A$ or ($x \in A$ and $x \in B$)

In either case, $x \in A$. Hence, $A \cup (A \cap B) \subseteq A$.

"$\supseteq$" Let $x \in A$. Then $x \in A$ or ($x \in A$ and $x \in B$) is true. So $x \in A \cup (A \cap B)$. Ergo, $A \subseteq A \cup (A \cap B)$ //

(c) $\boxed{\text{Thm}}$ $(A \cup B) \cap A^c = B - A$

$\boxed{\text{pf}}$ "$\subseteq$" Let $x \in (A \cup B) \cap A^c$. So $x \in A \cup B$ and $x \in A^c$

Thus, ($x \in A$ or $x \in B$) and $x \notin A$.

Since $x \notin A$ is true, $x \in B$ must be true.

So $x \in B$ and $x \notin A$. Hence, $x \in B - A$.

Ergo, $(A \cup B) \cap A^c \subseteq B - A$.

"$\supseteq$" Let $x \in B - A$. So $x \in B$ and $x \notin A$.

Thus ($x \in A$ or $x \in B$) is true and $x \in A^c$. So $x \in (A \cup B) \cap A^c$.

Therefore, $B - A \subseteq (A \cup B) \cap A^c$ //

(d) (i) $\boxed{\text{Thm}}$ $f(S \cap T) \subseteq f(S) \cap f(T)$

$\boxed{\text{pf}}$ Let $y \in f(S \cap T)$. So $y = f(x)$ for some $x \in S \cap T$.

Thus, $y = f(x)$ for some ($x \in S$ and $x \in T$)

Hence, by defn, $y \in f(S)$ and $y \in f(T)$.

Ergo $y \in f(S) \cap f(T)$, which gives $f(S \cap T) \subseteq f(S) \cap f(T)$ //

$\boxed{\text{Example}}$ $f(x) = x^2$, $S = \{-2, -3\}$

$T = \{-3, 2\}$

$S \cap T = \{-3\}$

$f(S \cap T) = f(\{-3\}) = \{9\}$

$f(S) = f(\{-2, -3\}) = \{4, 9\}$

$f(T) = f(\{-3, 2\}) = \{4, 9\}$

$f(S) \cap f(T) = \{4, 9\}$

NOT EQUAL

(ii) $\boxed{\text{Thm}}$ If f is an injection, then $f(S \cap T) = f(S) \cap f(T)$

$\boxed{\text{pf}}$ From (i), we know that $f(S \cap T) \subseteq f(S) \cap f(T)$.

Let $y \in f(S) \cap f(T)$. So $y \in f(S)$ and $y \in f(T)$.

Thus, $\exists x_1 \in S$ and $\exists x_2 \in T$ such that $y = f(x_1)$ and $y = f(x_2)$.

Since f is injective and $f(x_1) = f(x_2)$, we deduce that $x_1 = x_2$. So $x = x_1 = x_2$ is in $S \cap T$ and $y = f(x) \Rightarrow y \in f(S \cap T)$ //

$\boxed{3}$ (a) $\boxed{\text{Thm}}$ $\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, $\forall n \in \mathbb{N}$.

pf) Induction on n.

Base Step: For $n=1$, $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{6}{6}$ ✓

Inductive Step: Assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ for some $k \in \mathbb{N}$.

Then
$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= (k+1)^2 + \sum_{i=1}^{k} i^2 \\ &= (k+1)^2 + \frac{k(k+1)(2k+1)}{6} \quad \text{(by ind. hyp)} \\ &= [6(k+1)^2 + k(k+1)(2k+1)]/6 \\ &= (k+1)[6(k+1) + k(2k+1)]/6 \\ &= (k+1)[6k+6+2k^2+k]/6 \\ &= (k+1)[2k^2+7k+6]/6 \\ &= (k+1)(k+2)(2k+3)/6 \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad //\end{aligned}$$

(b) $\boxed{\text{Thm}}$ $\sum_{i=1}^{n} (2i-1) = 1+3+5+\cdots+(2n-1) = n^2$, $\forall n \in \mathbb{N}$.

pf) Induction on n.

Base Step: For $n=1$, $\sum_{i=1}^{1}(2i-1) = 1 = 1^2$. ✓

Ind. Step: Assume $\sum_{i=1}^{k}(2i-1) = k^2$ for some $k \in \mathbb{N}$.

Then
$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= (2(k+1)-1) + \sum_{i=1}^{k}(2i-1) \\ &= 2k+1 + k^2 \quad \text{(by ind. hyp.)} \\ &= (k+1)^2 \quad //\end{aligned}$$

(c) $\boxed{\text{Thm}}$ $24 \mid n(n^2-1)(3n+2)$ for all $n \in \mathbb{N}$.

pf) Note that $n(n^2-1)(3n+2) = n(n-1)(n+1)(3n+2)$.

First, we show that $n(n^2-1)(3n+2) \equiv 0 \pmod 8$.

If $n \in \mathbb{N}$, then $n \equiv 0,1,2,3,4,5,6$, or $7 \pmod 8$.

Checking these cases gives:

$0(0^2-1)(3(0)+2) \equiv 0 \pmod 8$ $\qquad$ $4(4^2-1)(3(4)+2) \equiv 0 \pmod 8$

0 $\quad 1(1^2-1)(3(1)+2) \equiv 0 \pmod 8$ $\qquad$ 24 $\quad 5(5^2-1)(3(5)+2) \equiv 0 \pmod 8$

8 $\quad 2(2^2-1)(3(2)+2) \equiv 0 \pmod 8$ $\qquad$ 120 $\quad 6(6^2-1)(3(6)+2) \equiv 0 \pmod 8$

8 $\quad 3(3^2-1)(3(3)+2) \equiv 0 \pmod 8$ $\qquad$ 48 $\quad 7(7^2-1)(3(7)+2) \equiv 0 \pmod 8$

Thus, $8 \mid n(n^2-1)(3n+2)$ $\forall n \in \mathbb{N}$.

Since $n(n-1)(n+1)$ are 3 consecutive integers, 3 must divide one of these, so $3 \mid n(n^2-1)(3n+2)$. ///

(d) $\boxed{\text{Thm}}$ $5^n + 5 < 5^{n+1}$, $\forall n \in \mathbb{N}$.

pf) Induction on n.

<u>Base Step</u>: For $n=1$, $5^1 + 5 = 10 < 25 = 5^{1+1}$. ✓

<u>Ind. Step</u>: Assume $5^k + 5 < 5^{k+1}$ for some $k \in \mathbb{N}$.

Then
$$
\begin{aligned}
5^{k+1} + 5 &= 5 \cdot 5^k + 5 \\
&< 5(5^{k+1} - 5) + 5 \quad \text{(by ind. hyp.)} \\
&= 5 \cdot 5^{k+1} - 20 \\
&< 5 \cdot 5^{k+1} = 5^{k+2} \quad //
\end{aligned}
$$

(e) $\boxed{\text{Thm}}$ If $1+x > 0$, then $(1+x)^n \geq 1 + nx$, $\forall n \in \mathbb{N}$.

pf) Induction on n.

<u>Base Step</u>: For $n=1$, $(1+x)^1 \geq 1 + 1\cdot x$. ✓

<u>Ind. Step</u>: Assume $(1+x)^k \geq 1 + kx$ for some $k \in \mathbb{N}$.

Then
$$
\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&\geq (1+x)(1+kx) \quad \text{(by ind. hyp.)} \\
&= 1 + kx + x + kx^2 \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x \quad \text{because } kx^2 \geq 0 \quad //
\end{aligned}
$$

$\boxed{4}$ (a) $f(x) = \dfrac{x}{x+1}$ injective?

Start with $f(x_1) = f(x_2)$
$$
\begin{aligned}
&\Rightarrow \frac{x_1}{x_1+1} = \frac{x_2}{x_2+1} \\
&\Rightarrow x_1(x_2+1) = x_2(x_1+1) \\
&\Rightarrow x_1x_2 + x_1 = x_1x_2 + x_2 \\
&\Rightarrow x_1 = x_2 \quad //
\end{aligned}
$$
$\boxed{\text{YES}}$

(b) (i) $h(x) = 2x$ $\qquad 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

But $2x = 1$ has no sol'n in $\mathbb{Z}$.

(ii) $h(x) = \begin{cases} \frac{1}{2}x, & x \text{ is even} \\ x, & x \text{ is odd.} \end{cases}$

If y is odd, then $h(y) = y$ ✓

If y is even, then $h(2y) = y$ ✓

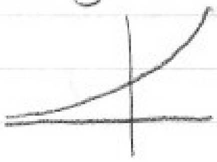
SURJECTIVE

NOT INJECTIVE BECAUSE $h(2) = h(1)$.

(iii) $h(x) = x$ is a bijection.

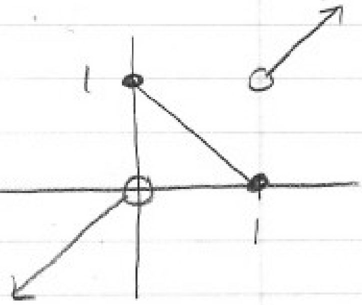
(c) (i) $f(x) = e^x$

$\boxed{\text{FALSE}}$



$\boxed{\text{UNBOUNDED NOT A SURJECTION.}}$

(ii) $f(x) = \begin{cases} x, & x < 0 \text{ OR } x > 1 \\ 1-x, & 0 \le x \le 1 \end{cases}$



$\boxed{\text{SURJECTIVE NOT MONOTONE}}$

$\boxed{\text{FALSE}}$

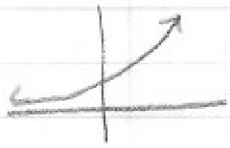
(iii) $f(x) = 7$ ← $\boxed{\text{MONOTONE NOT INJECTIVE}}$

$\boxed{\text{FALSE}}$

(iv) $f(x) = e^x$

$\boxed{\text{FALSE}}$

$\boxed{\text{STRICTLY MONOTONE NOT SURJECTIVE}}$



(d) (i) $\boxed{\text{TRUE}}$, pf) If $f(x_1) = f(x_2)$
then $g(f(x_1)) = g(f(x_1))$
because g is a function.
Hence $h(x_1) = h(x_2) \Rightarrow x_1 = x_2$
because h is injective. //

(ii) $\boxed{\text{FALSE}}$ $A = \{1\}$ $B = \{1,2\}$ $C = \{1\}$

$f(1) = 1$ $\underbrace{g(1) = 1, g(2) = 1}_{\boxed{\text{NOT INJECTIVE}}}$

$\underbrace{h(1) = g(f(1)) = 1}_{\boxed{\text{INJECTIVE}}}$

(iii) $\boxed{\text{FALSE}}$ SAME EXAMPLE ABOVE $\boxed{f \text{ is NOT SURJECTIVE}}$.

(iv) $\boxed{\text{TRUE}}$ pf) Let $y \in C$. Then $h(x) = y$ has a sol'n.
Since $y = h(x) = g(f(x))$ we see that
$g(z) = y$ has a sol'n (namely $z = f(x)$). //

(v) $\boxed{\text{TRUE}}$ pf ONE-TO-ONE: $h(x_1) = h(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ because g injective

$\Rightarrow x_1 = x_2$ because f injective.

ONTO: Let $y \in C$.

$g(z) = y$ has a sol'n with $z \in B$

Since g is onto.

$f(x) = z$ has a sol'n with $x \in A$

Since f is onto.

$\Rightarrow h(x) = g(f(x)) = y$. //

$\boxed{5}$ (a) $\boxed{\text{Thm}}$ $2 \mid n$, $2 \nmid m \Rightarrow 2 \nmid m+n$.

pf Since $2 \mid n$ and $2 \nmid m$,

$n = 2k$ and $m = 2\ell + 1$

for some $k, \ell \in \mathbb{Z}$.

Hence, $m+n = 2\ell + 1 + 2k = 2(k+\ell) + 1$.

So $2 \nmid m+n$. //

OR GIVE A PROOF BY CONTRADICTION.

That is, If $2 \mid m+n$, then $2 \mid n \Rightarrow 2 \mid m \rightarrow\leftarrow$ //

(b) $\boxed{\text{Thm}}$ If a, b are odd, then $8 \mid a^2 - b^2$.

pf Since a, b are odd, $\exists k, \ell \in \mathbb{Z}$ such that

$a = 2k+1$ and $b = 2\ell + 1$.

So $a^2 - b^2 = 4k^2 + 4k + 1 - 4\ell^2 - 4\ell - 1$

$= 4(k^2 + k - \ell^2 - \ell)$.

Since $k^2 + k = k(k+1)$, either k or $k+1$ is even

so $k(k+1)$ is even. Similarly, $\ell^2 + \ell$ is even.

So $2 \mid k^2 + k - \ell^2 - \ell \Rightarrow 8 \mid a^2 - b^2$. //