+ VERSION B: ELLIPTIC PARABOLOID

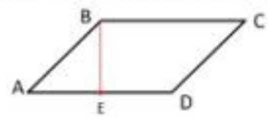
1. (12 pts)

(a) (3 pts) Consider $x^2 - 4y^2 - z = -9$.

- Give the 2D name of the traces when $x = k$ is fixed. Name: PARABOLAS
- Give the precise name of the 3D shape given by $x^2 - 4y^2 - z = -9$

Name: HYPERBOLIC PARABOLOID

(b) Consider the parallelogram show with $A(1,1,2)$, $B(2,3,7)$, $C(5,3,11)$, $D(4,1,6)$.



i. (4 pts) Find the area of the parallelogram ABCD.

$\vec{i}\ \vec{j}\ \vec{k}$

$AB = \langle 1, 2, 5 \rangle$

$AD = \langle 3, 0, 4 \rangle$

$= (8-0)\vec{i} - (4-15)\vec{j} + (0-6)\vec{k}$

$= \langle 8, 11, -6 \rangle$

CHECK: $8 + 22 - 30 = 0$ ✓

$24 + 0 - 24 = 0$ ✓

$\text{AREA} = |\langle 8, 11, -6 \rangle|$

$= \sqrt{64 + 121 + 36}$

$= \sqrt{221}$

≈ 14.87

Area = $\sqrt{221}$

ii. (3 pts) Find the vector of length 8 that points in the same direction as $\overrightarrow{BD}$.

5

$\overrightarrow{BD} = \langle 2, -2, -1 \rangle$

VERSION 2: 8

$\langle \frac{16}{3}, -\frac{16}{3}, -\frac{8}{3} \rangle$

$\frac{1}{\sqrt{4+4+1}} \langle 2, -2, -1 \rangle$ ← UNIT VECTOR

$\frac{5}{3} \langle 2, -2, -1 \rangle$

Vector: $\langle \frac{10}{3}, \frac{-10}{3}, -\frac{5}{3} \rangle$

iii. (2 pts) The line segment BE (shown) is perpendicular to the segment AD. Find the length of the segment AE.

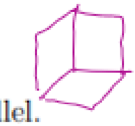
$$|AE| = \text{comp}_{\overrightarrow{AD}} \overrightarrow{AB} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{|\overrightarrow{AD}|} = \frac{3+0+20}{\sqrt{9+0+16}} = \frac{23}{5}$$

Length = $|AE|$ = $\frac{23}{5}$

2. (14 pts)

(a) (2 pts) True or false, circle one for each statement: VERSION B

- TRUE or FALSE : Two planes are always parallel or intersecting. SWAP



- TRUE or FALSE: Two planes perpendicular to a given plane must be parallel.

(b) (6 pts) Consider the line that contains the point $(5, 0, 0)$ and is orthogonal to the plane $3y - 4z = 10$. Find the two points of intersection of this line with the paraboloid $20x = y^2 + z^2$. (First find parametric equations for the line!)

$x = 5 + 0t$

$y = 0 + 3t$

$z = 0 - 4t$

INTERSECTION:

$20(5) = (3t)^2 + (-4t)^2$

$160 = 9t^2 + 16t^2$

$160 = 25t^2$

$t^2 = 4$

$t = \pm 2$ → $(5, 6, -8)$, $(5, -6, 8)$

Intersection Points: $(x, y, z) =$ $(5,6,-8)$, $(5,-6,8)$

(c) (6 pts) Find an equation for the plane that passes through the point A$(0, 0, 2)$ and contains the line of intersection of the planes $x + y - z = 1$ and $2x + y - 3z = -1$. And give the x-intercept of this new plane equation.

① $x + y - z = 1$
② $2x + y - 3z = -1$
$-x + 2z = 2$

$x = 0 \Rightarrow z = 1 \Rightarrow y = 2$ B$(0,2,1)$

$z = 0 \Rightarrow x = -2 \Rightarrow y = 3$ C$(-2,3,0)$

NOW WE HAVE 3 PTS! IT IS POSSIBLE TO FIND MANY OTHERS

$\vec{i}\ \vec{j}\ \vec{k}$

$\vec{AB} = \langle 0, 2, -1 \rangle$

$\vec{AC} = \langle -2, 3, -2 \rangle$

CHECK: $0 + 4 - 4 = 0$ ✓
$2 + 6 - 8 = 0$ ✓

$(-4 - -3)\vec{i} - (0 - -2)\vec{j} + (0 - -4)\vec{k} = \langle -1, 2, 4 \rangle$

x-intercept ⇒ $(?, 0, 0)$

$\Rightarrow -x + 0 + 4(0-2) = 0$

$-x - 8 = 0 \Rightarrow x = 8$

Plane Equation: $-x + 2y + 4(z-2) = 0$

x-intercept: $(x, y, z) =$ $(8, 0, 0)$

3. (12 pts)

(a) (6 pts) Find the angle of intersection of the curves: (Round final answer to the nearest degree)

- $\mathbf{r}_1(t) = \langle t, 3-t, t^4 \rangle$
- $\mathbf{r}_2(u) = \langle 2-u, 2u-2, 10-u^2 \rangle$

VERSION B

$7+u-u^2 \to \vec{r}_2{}'(u) = \langle -1, 2, 1-2u \rangle$

$\vec{r}_2{}'(3) = \langle -1, 2, -5 \rangle$

$\cos^{-1}\left(\dfrac{-1-2+20}{\sqrt{18}\sqrt{1+4+25}}\right)$

$\cos^{-1}\left(\dfrac{17}{\sqrt{18}\sqrt{30}}\right)$

$\approx 42.98°$

$\approx 43°$

① $t = 2-u$

② $3-t = 2u-2$

③ $t^4 = 10-u^2$

$3-(2-u) = 2u-2 \Rightarrow 1+u = 2u-2 \Rightarrow 3 = u \Rightarrow t = 2-3 = -1$

CHECK: $t^4 = (-1)^4 = 1$

$10-u^2 = 10-(3)^2 = 1$ ✓

$\vec{r}_1{}'(t) = \langle 1, -1, 4t^3 \rangle \qquad \vec{r}_1{}'(-1) = \langle 1, -1, -4 \rangle$

$\vec{r}_2{}'(u) = \langle -1, 2, -2u \rangle \qquad \vec{r}_2{}'(3) = \langle -1, 2, -6 \rangle$

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{-1-2+24}{\sqrt{1+1+16}\sqrt{1+4+36}} \qquad \theta = \cos^{-1}\left(\dfrac{21}{\sqrt{18}\sqrt{41}}\right)$

$\approx 39.37°$

Angle = 39°

(b) (6 pts) Let C be the curve of intersection of the surface $y = \frac{1}{2}x^2$ and the surface $z = \frac{1}{3}xy$. Parameterize this curve, then use the parameterization to give the arc length the curve from the point $(0, 0, 0)$ to $(3, \frac{9}{2}, 18)$

$x = t \qquad (0,0,0) \Rightarrow t = 0$

$y = \frac{1}{2}t^2 \qquad (3,9,18) \Rightarrow t = 3$

$z = \frac{1}{6}t^3$

$\int_0^3 \sqrt{(1)^2 + (t)^2 + (\tfrac{1}{2}t^2)^2}\, dt$

$= \int_0^3 \sqrt{1 + t^2 + \tfrac{1}{4}t^4}\, dt$

$= \int_0^3 \sqrt{\tfrac{1}{4}(4 + 4t^2 + t^4)}\, dt$

$= \frac{1}{2}\int_0^3 \sqrt{(t^2+2)^2}\, dt = \frac{1}{2}\int_0^3 t^2 + 2\, dt = \frac{1}{2}\left(\frac{1}{3}t^3 + 2t\right)\Big|_0^3$

$= \frac{1}{2}(9+6) = 15/2$

Arc Length = 15/2

4. (12 pts)

(a) (2 pts) True or false, circle one for each statement:

VERSION B

i. TRUE or FALSE : $\mathbf{r}'(t)$ and $\mathbf{N}(t)$ are always orthogonal.

SWAP

ii. TRUE or FALSE: $\mathbf{r}''(t)$ and $\mathbf{N}(t)$ are always parallel.

(b) Consider $\mathbf{r}(t) = \langle t^2, 3t+6, -2t^2 \rangle$.

i. (5 pts) Find the curvature at $t = 0$. (Give your answer as a decimal rounded to three digits)

$\vec{r}'(t) = \langle 2t, 3, -4t \rangle$

$\vec{r}''(t) = \langle 2, 0, -4 \rangle$

$\vec{r}'(0) = \langle 0, 3, 0 \rangle$

$\vec{r}''(0) = \langle 2, 0, -4 \rangle$

$(-12-0)\vec{i} - (0-0)\vec{j} + (0-6)\vec{k}$

$= \langle -12, 0, -6 \rangle$ CHECK: $0+0+0=0$ ✓ $-24+0+24=0$ ✓

$$\frac{|\vec{r}'(0) \times \vec{r}''(0)|}{|\vec{r}'(0)|^3} = \frac{\sqrt{144+0+36}}{\left(\sqrt{0+9+0}\right)^3} = \frac{\sqrt{180}}{3^3} = \frac{\sqrt{9}\sqrt{20}}{3^3} = \frac{3 \cdot 2\sqrt{5}}{3^3} = \frac{2\sqrt{5}}{9} \approx 0.4969$$

$\kappa(0) =$ 0.497

ii. (5 pts) Find the equation of the tangent line to $\mathbf{r}(t)$ at the point $(4, 12, -8)$ and find where this line intersects the xz-plane

$x = t^2 = 4$, $y = 3t+6 = 12$, $z = -2t^2 = -8 \Rightarrow t = 2$

$\vec{r}'(t) = \langle 2t, 3, -4t \rangle \Rightarrow \vec{r}'(2) = \langle 4, 3, -8 \rangle$

$x = 4 + 4t$
$y = 12 + 3t$
$z = -8 - 8t$

INTERSECT XZ-PLANE

$\Rightarrow 12 + 3t = 0 \Rightarrow t = \frac{-12}{3} = -4$

$\Rightarrow x = 4 - 16 = -12$
$y = 0$
$z = -8 + 32 = 24$

Intersection with xy-plane: $(x, y, z) =$ $(-12, 0, 24)$