1. (14 pts) **For ALL parts below**, consider the plane, $\mathcal{P}$, through A(0,0,1), B(1,1,3), and C(-1,2,4).

(a) To the nearest **degree**, find the angle at A in the triangle BAC.

$\vec{AB} = \langle 1-0, 1-0, 3-1 \rangle = \langle 1,1,2 \rangle$, $\vec{AC} = \langle -1-0, 2-0, 4-1 \rangle = \langle -1,2,3 \rangle$

$\underbrace{\langle 1,1,2 \rangle \cdot \langle -1,2,3 \rangle}_{-1+2+6} = \sqrt{1+1+2^2}\sqrt{1+4+9}\cos\theta$, $\theta = \cos^{-1}\left(\frac{7}{\sqrt{6}\sqrt{14}}\right) \approx \boxed{40^\circ}$

(b) Find the (x, y, z) point where $\mathcal{P}$ intersects the y-axis. (Hint: Find the equation for the plane).

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix} = (3-4)\vec{i} - (3--2)\vec{j} + (2--1)\vec{k}$$
$$= \langle -1, -5, 3 \rangle$$

PLANE : $-x - 5y + 3(z-1) = 0$

INTERSECTION WITH y-AXIS $\Leftrightarrow$ $x=0$ AND $z=0$

$-0 - 5y + 3(0-1) = 0 \Rightarrow -5y - 3 = 0 \Rightarrow y = -3/5$

$$\boxed{(0, -\tfrac{3}{5}, 0)}$$

(c) A particle starts at the point $(70, 0, 1)$ and moves toward the plane along a straight line that is orthogonal to the plane. At what point, (x, y, z), would this line intersect the plane?

LINE :
$$x = 70 - t$$
$$y = 0 - 5t$$
$$z = 1 + 3t$$

INTERSECT WITH PLANE

$$-(70-t) - 5(-5t) + 3(1+3t-1) \stackrel{?}{=} 0$$
$$-70 + t + 25t + 9t = 0$$
$$35t = 70 \qquad t = 2$$

$$\boxed{(x, y, z) = (68, -10, 7)}$$

2. (12 pts)

(a) Consider the surface containing all points that satisfy $x^2 + z^2 = 16 + y^2$.

i. Give the 2D names for the traces when the given variable is a constant k:

- For $x = k$, the traces are: HYPERBOLAS
- For $y = k$, the traces are: CIRCLES
- ~~For $z = k$, the traces are: HYPERBOLAS~~

ii. Using the precise name from class, give the 3D name of this shape.

HYPERBOLOID OF ONE SHEET

iii. A particle is traveling **on** the surface $x^2 + z^2 = 16 + y^2$ in such a way that y is negative, $x = 5\cos(t)$ and $z = 5\sin(t)$ and for all times t. Find the value, or formula, for the y-coordinate of the particle at all times t.

$$(5\cos(t))^2 + (5\sin(t))^2 \stackrel{?}{=} 16 + y^2$$

$$25 = 16 + y^2 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3 \qquad \boxed{y = -3}$$

(b) Consider the curve given by the polar equation $r = 6\cos(\theta)$.

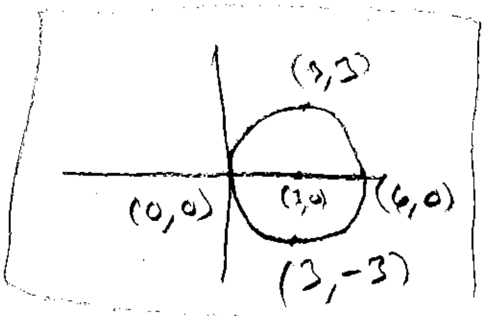
i. Convert to Cartesian coordinates, then draw a rough sketch of it in the xy-plane with several points labeled. (Hint: It is a shape you know well).

$$r = 6\,x/r \Rightarrow r^2 = 6x \Rightarrow x^2 + y^2 = 6x$$

$$\Rightarrow x^2 - 6x + y^2 = 0$$

$$x^2 - 6x + 9 + y^2 = 9$$

$$(x-3)^2 + y^2 = 9$$



CIRCLE CENTERED AT (3,0) OF RADIUS 3

ii. There are two Cartesian points on this graph where the tangent lines are horizontal. Find polar coordinates (r, θ) for these points. (Hint: You can use the graph.).

$$(3,3) \Rightarrow \theta = \pi/4 \Rightarrow r = 6\cos(\pi/4) = 6\frac{\sqrt{2}}{2} = 3\sqrt{2}$$

$$(3,-3) \Rightarrow \theta = -\pi/4 \Rightarrow r = 3\sqrt{2}$$

on $+2\pi k$

$$(r,\theta) = \boxed{(3\sqrt{2}, \pi/4) \quad \text{or} \quad (3\sqrt{2}, -\pi/4)}$$

on $+2\pi k$

3. (13 pts) **For ALL parts on this page**, two particles travel along curves given by

$$\mathbf{r}_1(t) = \langle 2t, 3t^2, 2t^3 \rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle 2-2t, 3+3t, 2-6t \rangle,$$

where t is time in seconds and distances are in feet.

(a) Find a vector that is **tangent** to $\mathbf{r}_1(t)$ at $t = 1$ and has length 10.

$\vec{r}_1'(t) = \langle 2, 6t, 6t^2 \rangle$

$\vec{r}_1'(1) = \langle 2, 6, 6 \rangle$

$|\vec{r}_1'(1)| = \sqrt{2^2+6^2+6^2} = \sqrt{4+36+36} = \sqrt{76}$

$\boxed{\frac{10}{\sqrt{76}} \langle 2, 6, 6 \rangle}$

or negative

(b) Consider $\mathbf{r}_2(t)$ which starts at $(2, 3, 2)$.

i. Find and simplify the arc length function, $s = s(t) = \int_0^t |\mathbf{r}_2'(u)| du$.

$$S = \int_0^t \sqrt{(-2)^2+(3)^2+(-6)^2}\, du = \int_0^t \sqrt{4+9+36}\, du = \int_0^t 7\, du$$

$$\boxed{S = 7t}$$

ii. The particle stops at the instant it has traveled 28 feet from its starting location. Give its (x, y, z) coordinates at this instant.

$S = 28 \Rightarrow t = 4 \Rightarrow (x,y,z) = (2-2(4), 3+3(4), 2-6(4))$

$= \boxed{(-6, 15, -22)}$

(c) Find the (x, y, z) point(s) at which the **paths** of the two particles intersect. (This is NOT a collision question).

$2t = 2-2u \Rightarrow t = 1-u$

$3t^2 = 3+3u \Rightarrow t^2 = 1+u$

$2t^3 = 2-6u$

$(1-u)^2 = 1+u$

$1-2u+u^2 = 1+u$

$u^2-3u = 0$

$u(u-3) = 0$

$u=0 \Downarrow t=1$ or $u=3 \Downarrow t=-2$

$u=0, t=1 \Rightarrow \boxed{(x,y,z) = (2,3,2)}$

$u=3, t=-2 \Rightarrow \boxed{(x,y,z) = (-4,12,-16)}$

4. (10 pts) **NOTE: The two parts below are NOT related!**

(a) Find all values of t at which the tangent line to the curve $x = 8 - t^3$, $y = 42t - 10t^2$ is orthogonal to the vector $\langle 2, -3 \rangle$.

$\vec{r}\,'(t) = \langle -3t^2, 42 - 20t \rangle =$ a tangent vector at t.

WANT ORTHOGONAL TO $\langle 2, -3 \rangle$

$\langle 2, -3 \rangle \cdot \langle -3t^2, 42 - 20t \rangle \overset{?}{=} 0$

$-6t^2 - 126 + 60t = 0$

$t^2 - 10t + 21 = 0$

$(t-3)(t-7) = 0$

$\boxed{t = 3 \text{ or } t = 7}$

(b) A small bug is moving according to the vector function

$$\mathbf{r}(t) = \langle t\sin(\pi t), \ln(t), t^2 - 4e^{(2-2t)} \rangle.$$

At time $t = 1$ the bug leaves the curve and follows the path of the tangent line. Find the (x, y, z) coordinates where the bug's tangent line path would intersect the xy-plane.

$\vec{r}(1) = \langle 0, 0, 1 - 4 \rangle = \langle 0, 0, -3 \rangle$

$\vec{r}\,'(t) = \langle \sin(\pi t) + \pi t\cos(\pi t), \frac{1}{t}, 2t + 8e^{(2-2t)} \rangle$

$\vec{r}\,'(1) = \langle 0 - \pi, 1, 2 + 8 \rangle = \langle -\pi, 1, 10 \rangle$

TANGENT LINE: $x = 0 - \pi t$, $y = 0 + t$, $z = -3 + 10t$

INTERSECT XY-PLANE: $-3 + 10t \overset{?}{=} 0 \Rightarrow t = 3/10$

$\boxed{(x, y, z) = \left(-\frac{3\pi}{10}, \frac{3}{10}, 0\right)}$