1. (10 points) Consider the triangle determined by the points $P(1,0,-2)$, $Q(0,2,3)$, and $R(1,1,1)$.

(a) (4 pts) Find the equation of the plane through P, Q, and R.
(Give your answer in the scalar form $ax+by+cz+d=0$.)

$\vec{PQ} = \langle 0,2,3\rangle - \langle 1,0,-2\rangle = \langle -1,2,5\rangle$

$\vec{PR} = \langle 1,1,1\rangle - \langle 1,0,-2\rangle = \langle 0,1,3\rangle$

$$\begin{array}{cccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 5 & -1 & 2 & 5 \\ 0 & 1 & 3 & 0 & 1 & 3 \end{array}$$

$\vec{PQ} \times \vec{PR} = \langle 6-5, 0--3, -1-0\rangle = \langle 1,3,-1\rangle$

PLANE: $\langle 1,3,-1\rangle \cdot (\langle x,y,z\rangle - \langle 1,0,-2\rangle) = 0$

$(x-1) + 3(y-0) - (z+2) = 0$

$\boxed{x + 3y - z - 3 = 0}$ OR ANY NONZERO CONSTANT MULTIPLE OF THIS EQUATION

(b) (4 pts) Find the angle of the triangle at Q.
That is, find the angle between the two vectors $\vec{QP}$ and $\vec{QR}$.
(Give your final answer as a decimal to the nearest degree.)

$\vec{QP} = \langle 1,-2,-5\rangle$ $\quad$ $\vec{QR} = \langle 1,1,1\rangle - \langle 0,2,3\rangle = \langle 1,-1,-2\rangle$

$\vec{QP} \cdot \vec{QR} = |\vec{QP}||\vec{QR}|\cos(\theta)$

$1 + 2 + 10 = \sqrt{1+4+25}\sqrt{1+1+4}\cos(\theta)$

$13 = \sqrt{30}\sqrt{6}\cos(\theta)$

$13 = \sqrt{180}\cos(\theta)$

$\cos(\theta) = \dfrac{13}{\sqrt{180}} = \dfrac{13}{6\sqrt{5}}$

$\theta = \cos^{-1}\left(\dfrac{13}{6\sqrt{5}}\right)$

≈ 14.31227548 deg

$\approx \boxed{14 \text{ degrees}}$

(c) (2 pts) Find the area of the triangle.

$\text{AREA} = \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{1+9+1} = \boxed{\frac{1}{2}\sqrt{11}}$

≈ 1.658312

2. (10 points)

(a) (4 pts) Find parametric equations for the line through the point (1,0,2) and orthogonal to the plane $3x - 2y + z = 4$.

POINT: $\langle x_0, y_0, z_0 \rangle = \langle 1, 0, 2 \rangle$

DIRECTION VECTOR: $\langle a, b, c \rangle = \langle 3, -2, 1 \rangle$

LINE: $\langle x, y, z \rangle = \langle 1, 0, 2 \rangle + t \langle 3, -2, 1 \rangle$

$$\boxed{\begin{aligned} x &= 1 + 3t \\ y &= -2t \\ z &= 2 + t \end{aligned}}$$

(b) (6 pts) Find all points where the sphere $x^2 + y^2 - 2y + z^2 = 8$ intersects the line through the points (0,1,0) and (8,5,1).

SPHERE: $x^2 + y^2 - 2y + 1 + z^2 = 9$

$x^2 + (y-1)^2 + z^2 = 9$

LINE: pt: $\langle x_0, y_0, z_0 \rangle = \langle 0, 1, 0 \rangle$ direction: $\langle a, b, c \rangle = \langle 8, 5, 1 \rangle - \langle 0, 1, 0 \rangle$

$= \langle 8, 4, 1 \rangle$

$\langle x, y, z \rangle = \langle 0, 1, 0 \rangle + t \langle 8, 4, 1 \rangle$

$x = 8t, \quad y = 1 + 4t, \quad z = t$

INTERSECTION: $(8t)^2 + (1 + 4t - 1)^2 + (t)^2 = 9$

$64t^2 + 16t^2 + t^2 = 9$

$81t^2 = 9$

$t^2 = 1/9 \qquad t = \pm 1/3$

POINTS: $(x, y, z) = \left(-\frac{8}{3}, 1 - \frac{4}{3}, -\frac{1}{3}\right) = \left(-\frac{8}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$

AND $(x, y, z) = \left(\frac{8}{3}, 1 + \frac{4}{3}, \frac{1}{3}\right) = \left(\frac{8}{3}, \frac{7}{3}, \frac{1}{3}\right)$

3. (10 points)

(a) (6 pts) Find the equation of the tangent line, namely $y = mx + b$, to the polar curve $r = 3 - 3\sin(\theta)$ at $\theta = \pi$.

$$\frac{dr}{d\theta} = -3\cos(\theta) \qquad \frac{dr}{d\theta}(\pi) = -3\cos(\pi) = 3$$

$$r(\pi) = 3 - 3\sin(\pi) = 3$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$ at $\theta = \pi$ we get $\frac{dy}{dx} = \frac{3\cdot 0 + 3(-1)}{3\cdot(-1) - 3\cdot 0}$

$\frac{dy}{dx} = 1 =$ slope

$y = x + b$

$x = r\cos(\theta)$ at $\theta = \pi$: $x = 3\cos(\pi) = -3$

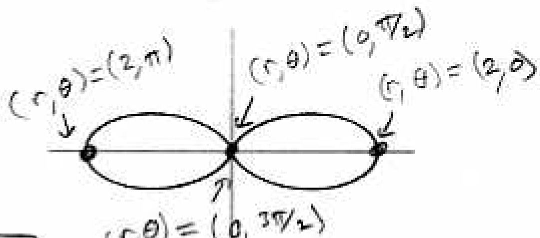
$y = r\sin(\theta)$: $y = 3\sin(\pi) = 0$

$0 = -3 + b \Rightarrow b = 3$

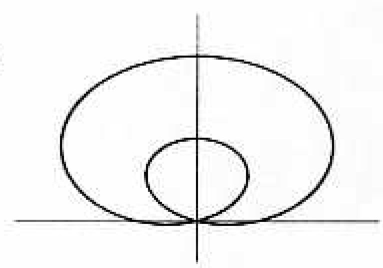
$\boxed{y = x + 3}$

(b) (4 pts) Match the polar equations to the correct graphs. Put the number of the equation next to the correct picture in the blanks provided. No formal explanation is required. (There is one extra picture that won't be labeled).

1. $r = 3 - 3\sin(\theta)$
2. $r = 1 + \cos(2\theta)$
3. $r = \sin(3\theta)$



$\boxed{2}$



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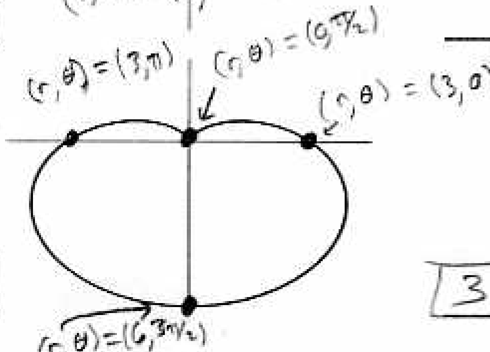
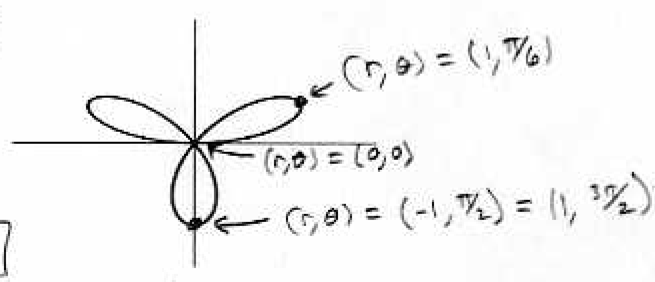


$\boxed{1}$



$\boxed{3}$

①

θ	r
0	3
$\pi/2$	0
π	3
$3\pi/2$	6
2π	3

②

θ	r
0	2
$\pi/2$	0
π	2
$3\pi/2$	0
2π	2

③

θ	r
0	0
$\pi/2$	-1
π	0
$3\pi/2$	1
2π	0

θ	r
$\pi/6$	1
$\pi/3$	0

4. (10 points) For $t > 0$, a bumblebee is traveling along the curve given by the vector function $\mathbf{r}(t) = \langle \frac{8}{t}, 12\sqrt{t}, t^2 \rangle$.

(a) (4 pts) Find the unit tangent at $t = 2$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

$$\vec{r}\,'(t) = \langle -\frac{8}{t^2}, \frac{12}{2\sqrt{t}}, 2t \rangle$$

$$\vec{r}\,'(2) = \langle -\frac{8}{4}, \frac{6}{\sqrt{2}}, 4 \rangle$$

$$\vec{r}\,'(2) = \langle -2, 3\sqrt{2}, 4 \rangle$$

$$\vec{T}(2) = \frac{\langle -2, 3\sqrt{2}, 4 \rangle}{\sqrt{4 + 9 \cdot 2 + 16}} = \frac{\langle -2, 3\sqrt{2}, 4 \rangle}{\sqrt{38}}$$

$$\vec{T}(2) = \langle -\frac{2}{\sqrt{38}}, \frac{3\sqrt{2}}{\sqrt{38}}, \frac{4}{\sqrt{38}} \rangle$$

(b) (6 pts) Find parametric equations for the tangent line to the curve given by $\mathbf{r}(t)$ at the point $(2, 24, 16)$.

$x = \frac{8}{t} = 2 \Rightarrow 8 = 2t \Rightarrow \boxed{t = 4}$

check: $y = 12\sqrt{t} = 24 \Rightarrow \sqrt{t} = 2 \Rightarrow \boxed{t = 4}$

$z = t^2 = 16 \Rightarrow \boxed{t = \pm 4}$

$t = 4$

POINT: $\langle 2, 24, 16 \rangle$

DIRECTION: $\vec{r}\,'(4) = \langle -\frac{8}{4^2}, \frac{12}{2\sqrt{4}}, 2 \cdot 4 \rangle$

$= \langle -\frac{1}{2}, 3, 8 \rangle$

LINE: $\langle x, y, z \rangle = \langle 2, 24, 16 \rangle + t\langle -\frac{1}{2}, 3, 8 \rangle$

$$\boxed{\begin{aligned} x &= 2 - \tfrac{1}{2}t \\ y &= 24 + 3t \\ z &= 16 + 8t \end{aligned}}$$

5. (10 points) A baseball player named Franklin is running on a coordinate system. His location (x, y) at time t seconds is given by the parametric equations

$$x = 6 + t^3,\ y = 5 + \frac{3}{2}t^2.$$

(a) (4 pts) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = 1$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t}{3t^2} = t^{-1}$$

$\frac{dy}{dx}$ at $t=1$ is $\boxed{1}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(dy/dx) = \frac{d/dt\,(dy/dx)}{dx/dt} = \frac{-t^{-2}}{3t^2} = -\frac{1}{3}t^{-4}$$

$\frac{d^2y}{dx^2}$ at $t=1$ is $\boxed{-\frac{1}{3}}$

(b) (6 pts) Franklin runs along the parametric curve from the point (6,5) to the point (14,11). Find the distance traveled along the curve between these two points. (Set up and evaluate)

$(x, y) = (6,5) \Leftrightarrow 6 = 6 + t^3,\ 5 = 5 + \frac{3}{2}t^2 \Leftrightarrow t = 0$

$(x, y) = (14,11) \Leftrightarrow 14 = 6 + t^3,\ 11 = 5 + \frac{3}{2}t^2 \Leftrightarrow t = 2$

$8 = t^3,\quad 6 = \frac{3}{2}t^2$

$t = 2,\quad 4 = t^2$

$t = \pm 2$

$$\text{ARC LENGTH} = \int_0^2 \sqrt{(3t^2)^2 + (3t)^2}\,dt$$

$$= \int_0^2 \sqrt{9t^4 + 9t^2}\,dt$$

$$= \int_0^2 3t\sqrt{t^2+1}\,dt$$

$u = t^2 + 1$

$du = 2t\,dt \qquad dt = \frac{du}{2t}$

$t = 0 \Leftrightarrow u = 1$

$t = 2 \Leftrightarrow u = 5$

$$= 3\int_1^5 \cancel{t}\sqrt{u}\,\frac{du}{2\cancel{t}}$$

$$= \frac{3}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^5 = \boxed{5^{3/2} - 1} \approx 10.18033989$$