1. (14 pts) **For ALL parts below**, consider the plane, $\mathcal{P}$, through A(0,0,1), B(1,1,3), and C(-1,2,4).

(a) To the nearest **degree**, find the angle at A in the triangle BAC.

$$\vec{AB} = \langle 1-0, 1-0, 3-1 \rangle = \langle 1,1,2 \rangle \quad , \quad \vec{AC} = \langle -1-0, 2-0, 4-1 \rangle = \langle -1,2,3 \rangle$$

$$\underbrace{\langle 1,1,2\rangle \cdot \langle -1,2,3\rangle}_{-1+2+6} = \sqrt{1+1+2^2}\,\sqrt{1+4+9}\,\cos\theta \quad , \quad \theta = \cos^{-1}\left(\frac{7}{\sqrt{6}\sqrt{14}}\right) \approx \boxed{40^\circ}$$

(b) Find the (x, y, z) point where $\mathcal{P}$ intersects the y-axis. (Hint: Find the equation for the plane).

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix} = (3-4)\vec{i} - (3--2)\vec{j} + (2--1)\vec{k}$$
$$= \langle -1, -5, 3 \rangle$$

PLANE : $-x - 5y + 3(z-1) = 0$

INTERSECTION WITH y-AXIS $\iff$ $x=0$ AND $z=0$

$$-0 - 5y + 3(0-1) = 0 \Rightarrow -5y - 3 = 0 \Rightarrow y = -3/5$$

$$\boxed{\left(0, -\tfrac{3}{5}, 0\right)}$$

(c) A particle starts at the point $(70, 0, 1)$ and moves toward the plane along a straight line that is orthogonal to the plane. At what point, (x, y, z), would this line intersect the plane?

LINE :
$$x = 70 - t$$
$$y = 0 - 5t$$
$$z = 1 + 3t$$

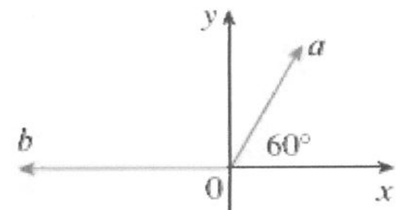
INTERSECT WITH PLANE

$$-(70-t) - 5(-5t) + 3(1+3t-1) \stackrel{?}{=} 0$$
$$-70 + t + 25t + 9t = 0$$
$$35t = 70 \qquad t = 2$$

$$\boxed{(x,y,z) = (68, -10, 7)}$$

1. (11 points)

(a) The forces **a** and **b** are the pictured. If $|\mathbf{a}| = 80$ N and $|\mathbf{b}| = 100$ N, find the angle the **resultant** force makes with the positive x-axis.
(Give your answer rounded to the nearest degree).



$\vec{a} = \langle 80\cos(60°), 80\sin(60°)\rangle = \langle 40, 40\sqrt{3}\rangle$

$\vec{b} = \langle -100, 0\rangle$

RESULTANT FORCE $= \vec{a} + \vec{b}$

$= \langle -60, 40\sqrt{3}\rangle = \vec{u}$

ANGLE WITH POSITIVE X-AXIS? CAN USE $\vec{v} = \langle 1, 0\rangle$ AND $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$\Rightarrow -60 = \sqrt{60^2 + 40^2\cdot 3}\,\sqrt{1}\,\cos\theta$

$-60 = \sqrt{8400}\,\cos\theta$

$\cos(\theta) = \frac{-60}{\sqrt{8400}} = \frac{-60}{20\sqrt{21}} = -\frac{3}{\sqrt{21}} = -\sqrt{\frac{3}{7}} = -\frac{\sqrt{21}}{7}$

$\theta = \cos^{-1}\left(\frac{-60}{\sqrt{8400}}\right) \approx 130.9933946 \approx \boxed{131°}$

(2.2845 radians)

(b) Find the center and radius of the sphere with points $P(x, y, z)$ such that the distance from P to $A(0, 0, 2)$ is triple the distance from P to $B(0, 0, 0)$.

$\sqrt{x^2 + y^2 + (z-2)^2} = 3\sqrt{x^2 + y^2 + z^2}$

$x^2 + y^2 + z^2 - 4z + 4 = 9x^2 + 9y^2 + 9z^2$

$4 = 8x^2 + 8y^2 + 8z^2 + 4z$

$\frac{1}{16} + \frac{1}{2} = x^2 + y^2 + z^2 + \frac{1}{2}z + \frac{1}{16}$

$\hookrightarrow \frac{1}{4}$

$\frac{9}{16} = x^2 + y^2 + (z + \frac{1}{4})^2$

$\boxed{\text{CENTER} = (0, 0, -\tfrac{1}{4}) \qquad \text{RADIUS} = \tfrac{3}{4}}$

1. (11 pts)

5 (a) Find parametric equations for the line of intersection of the two planes $2x - y + 8z = 14$ and $2x - 2y + 4z = 2$.

$2x = 14 + y - 8z$

COMBINING $\Rightarrow$ $y + 4z = 12$

ONE POINT $\left[\begin{array}{l} y = 0 \Rightarrow z = 3 \Rightarrow 2x = 14 + 0 - 8(3) = -10 \Rightarrow x = -5 \\ P(-5, 0, 3) \end{array} \right.$

ANOTHER POINT $\left[\begin{array}{l} z = 0 \Rightarrow y = 12 \Rightarrow 2x = 14 + 12 - 8(0) = 26 \Rightarrow x = 13 \\ Q(13, 12, 0) \end{array} \right.$

$\vec{PQ} = \langle 18, 12, -3 \rangle$

$$\boxed{\begin{aligned} x &= -5 + 18t \\ y &= 0 + 12t \\ z &= 3 - 3t \end{aligned}}$$

MANY CORRECT ANSWERS

① (x_0, y_0, z_0) NEEDS TO BE ON BOTH PLANES

② DIRECTION VECTOR MUST BE PARALLEL TO $\langle 18, 12, -3 \rangle$

3 (b) Consider the curve $y = 10 + 4x - x^2$ at $(x, y) = (3, 13)$.

i. Find a vector, $\mathbf{v}$, that has length 4 and is parallel to the tangent line to $y = 10 + 4x - x^2$ at $x = 3$.

$y' = 4 - 2x$

$y'(3) = 4 - 2(3) = -2$

SO $\langle 1, -2 \rangle$ IS PARALLEL TO THE TANGENT

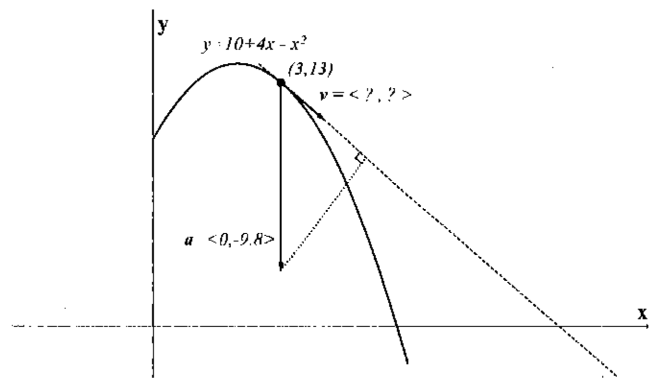
$$\boxed{\frac{4}{\sqrt{5}} \langle 1, -2 \rangle = \left\langle \frac{4}{\sqrt{5}}, \frac{-8}{\sqrt{5}} \right\rangle}$$

OR

$$\boxed{-\frac{4}{\sqrt{5}} \langle 1, -2 \rangle = \left\langle \frac{-4}{\sqrt{5}}, \frac{8}{\sqrt{5}} \right\rangle}$$

3 ii. Find the length of the projection of $\mathbf{a} = \langle 0, -9.8 \rangle$ onto $\mathbf{v}$, from part (i).

$$\text{comp}_{\vec{v}} \langle 0, -9.8 \rangle = \frac{0 + \frac{4}{\sqrt{5}} \cdot 19.6}{\underbrace{|\vec{v}|}_{4}} = \boxed{\frac{19.6}{\sqrt{5}} \approx 8.765}$$



IF YOU USE $\vec{v} = \left\langle -\frac{4}{\sqrt{5}}, \frac{8}{\sqrt{5}} \right\rangle$, THEN YOU GET $-\frac{19.6}{\sqrt{5}}$ WHICH IS ALSO ACCEPTED IF IT MATCHES YOUR CHOICE OF $\vec{v}$.

1. (13 pts)

(a) Find a vector that has length 7 and is orthogonal to both $\mathbf{u} = \langle 1, 0, 2 \rangle$ and $\mathbf{v} = \langle 3, -2, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 3 & -2 & 1 \end{vmatrix} = (0 - -4)\vec{i} - (1-6)\vec{j} + (-2-0)\vec{k} = \langle 4, 5, -2 \rangle$$

CHECK: $4 + 0 - 2 = 0$ ✓
$12 - 10 - 2 = 0$ ✓

$$|\vec{u} \times \vec{v}| = \sqrt{16+25+4} = \sqrt{45} = 3\sqrt{5}$$

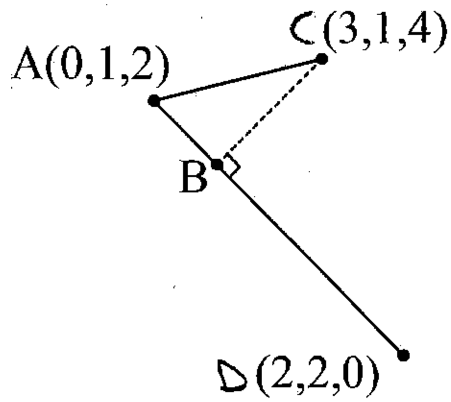
$$\boxed{\frac{7}{\sqrt{45}} \langle 4, 5, -2 \rangle} \quad \text{OR} \quad \boxed{\frac{-7}{\sqrt{45}} \langle 4, 5, -2 \rangle}$$

(b) Find the distance from point A to point B in the picture below (Hint: Use vector tools!)

$\vec{AC} = \langle 3, 0, 2 \rangle$

$\vec{AD} = \langle 2, 1, -2 \rangle$

$$\text{COMP}_{\vec{AD}} \vec{AC} = \frac{6+0-4}{\sqrt{4+1+4}} = \boxed{\frac{2}{3}}$$



(c) Consider the line through the points $(0, 0, 1)$ and $(3, 4, 5)$. Find the (x, y, z) point(s) where the line intersects the cylinder $x^2 + y^2 = 4$.

LINE: $x = 0 + 3t, \; y = 0 + 4t, \; z = 1 + 4t$

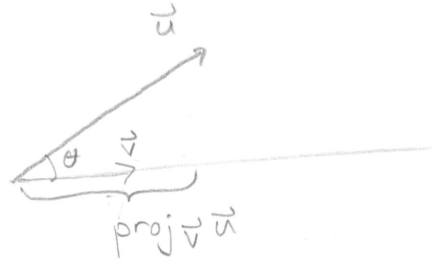
INTERSECTION: $(3t)^2 + (4t)^2 = 4 \Rightarrow 25t^2 = 4 \Rightarrow t^2 = \frac{4}{25}$

$\Rightarrow t = \pm \frac{2}{5}$

$t = -\frac{2}{5} \Rightarrow$ $(x,y,z) = \left(-\frac{6}{5}, -\frac{8}{5}, 1 - \frac{8}{5}\right) = \left(-\frac{6}{5}, -\frac{8}{5}, -\frac{3}{5}\right)$

$t = \frac{2}{5} \Rightarrow$ $(x,y,z) = \left(\frac{6}{5}, \frac{8}{5}, 1 + \frac{8}{5}\right) = \left(\frac{6}{5}, \frac{8}{5}, \frac{13}{5}\right)$

2. Consider the vectors $\mathbf{u} = \langle 3, -2, 5\rangle$, $\mathbf{v} = \langle 2, -1, 0\rangle$.



(a) (4 pts) Find the vector obtained by projecting **u** onto **v**.

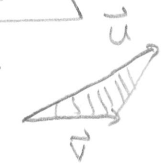
DESIRED LENGTH $= |\vec{u}| \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$

IN DIRECTION OF THE UNIT VECTOR $\dfrac{1}{|\vec{v}|} \vec{v}$

THUS,

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{6+2+0}{4+1+0} \langle 3, -1, 0\rangle = \boxed{\frac{8}{5} \langle 2, -1, 0\rangle = \langle \tfrac{16}{5}, -\tfrac{8}{5}, 0\rangle}$$

(b) (4 pts) Find the area of the triangle with corners $(0, 0, 0)$, $(3, -2, 5)$ and $(2, -1, 0)$.



$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 5 \\ 2 & -1 & 0 \end{vmatrix} = (0 - -5)\vec{i} - (0 - 10)\vec{j} + (-3 - -4)\vec{k}$$
$$= \langle 5, 10, 1\rangle$$

AREA OF TRIANGLE $= \frac{1}{2} |\vec{u} \times \vec{v}| = \frac{1}{2}\sqrt{25 + 100 + 1}$

$= \boxed{\frac{1}{2}\sqrt{126}}$ units^2

3. (7 pts) Find the angle of intersection of the two curves:

$$\mathbf{r}_1(t) = \langle t, 2 - t, t^2 - 5t - 11\rangle \text{ and } \mathbf{r}_2(u) = \langle 5 - 2u, u - 4, u^3 + 4\rangle.$$

(Give your answer in degrees rounded to two digits after the decimal).

$\boxed{\text{INTERSECT}}$

$\left.\begin{array}{l} ① \ t = 5 - 2u \\ ② \ 2 - t = u - 4 \end{array}\right\}$ ① & ② $\Rightarrow$ $2 - (5 - 2u) = u - 4$

$-3 + 2u = u - 4$

$u = -1 \Rightarrow t = 7$

③ $t^2 - 5t - 11 = 49 - 35 - 11 = 3$ ✓

$u^3 + 4 = 3$ ✓

$\boxed{\text{DIRECTIONS}}$ $\vec{r}_1{}'(t) = \langle 1, -1, 2t - 5\rangle$ $\quad \vec{r}_1{}'(7) = \langle 1, -1, 9\rangle$

$\vec{r}_2{}'(u) = \langle -2, 1, 3u^2\rangle$ $\quad \vec{r}_2{}'(-1) = \langle -2, 1, 3\rangle$

$\boxed{\text{ANGLE}}$ $\langle 1, -1, 9\rangle \cdot \langle -2, 1, 3\rangle = \sqrt{1 + 1 + 81}\sqrt{4 + 1 + 9} \cos\theta$

$$\cos\theta = \frac{-2 - 1 + 27}{\sqrt{83}\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{24}{\sqrt{83}\sqrt{14}}\right) \approx 45.24654254^\circ$$

$\boxed{45.25^\circ}$