1. (12 points)

(a) Find the equation of the plane that goes through the three points $\overset{A}{(1,2,5)}$, $\overset{B}{(2,2,2)}$, $\overset{C}{(3,3,3)}$.

$\vec{AB} = \langle 1, 0, -3 \rangle$

$\vec{AC} = \langle 2, 1, -2 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -3 \\ 2 & 1 & -2 \end{vmatrix} = (0 - -3)\vec{i} - (-2 - -6)\vec{j} + (1 - 0)\vec{k}$$

$= \langle 3, -4, 1 \rangle$ ← check dot products ✓

NORMAL

ANY POINT

$$3(x-1) - 4(y-2) + (z-5) = 0$$

$$3x - 3 - 4y + 8 + z - 5 = 0$$

$$\boxed{3x - 4y + z = 0}$$

(b) Find parametric equations for the line of intersection of $x - z \overset{①}{=} 10$ and $x + y + 2z \overset{②}{=} 0$.

FIND TWO POINTS: $x = 0 \Rightarrow z = -10$ in ①

① & ② $\Rightarrow (0) + y + 2(-10) = 0 \Rightarrow y = 20$

$P(0, 20, -10)$

$z = 0 \Rightarrow x = 10$ in ①

① & ② $\Rightarrow (10) + y + 2(0) = 0 \Rightarrow y = -10$

$Q(10, -10, 0)$

DIRECTION: $\vec{v} = \vec{PQ} = \langle 10, -30, 10 \rangle$

$$\boxed{\begin{aligned} x &= 0 + 10t \\ y &= 20 - 30t \\ z &= -10 + 10t \end{aligned}}$$

ANY POINT ON LINE

ANY VECTOR PARALLEL TO $\langle 10, -30, 10 \rangle$

2. (5 pts) Consider the surface $z^2 - 6x^2 - 6y^2 = -9$.

(a) (2 pts) Give the precise name of this surface as given in the book and in class.

multiply by -1 to get $-z^2 + 6x^2 + 6y^2 = 9$ → also traces hyp/hyp/circle ← always defined

$\boxed{\text{HYPERBOLOID OF ONE SHEET}}$

(b) (3 pts) The surfaces $z^2 - 6x^2 - 6y^2 = -9$ and $z = x^2 + y^2$ intersect to form a circle that is parallel to the xy-plane. Find the center and radius of this circle. (For the center, give the (x, y, z) coordinates).

$z^2 - 6(x^2+y^2) = -9$ AND $x^2+y^2 = z$ combine to give

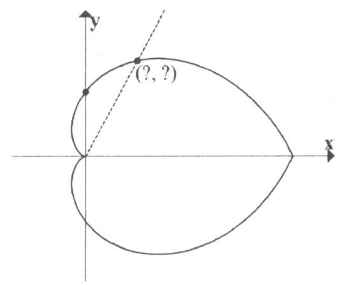
$z^2 - 6z = -9 \Rightarrow z^2 - 6z + 9 = 0 \Rightarrow (z-3)^2 = 0 \Rightarrow \boxed{z=3}$

$x^2+y^2 = 3$

$\boxed{\begin{array}{l}\text{CENTER}: (0,0,3) \\ \text{RADIUS}: \sqrt{3}\end{array}}$

3. (8 pts) Consider the polar curve $r = 1 - \sin\left(\frac{1}{2}\theta\right)$. (shown below)

(a) (3 pts) In the first quadrant, the line $y = \sqrt{3}x$ intersects the curve at the origin and at one other point (as shown). Find this other point. (Note: The line makes a 60 degree angle with the positive x-axis).



$\theta = \pi/3 \qquad r = 1 - \sin\left(\frac{1}{2}\frac{\pi}{3}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

$\boxed{\begin{array}{l} x = \frac{1}{2}\cos(\pi/3) = \frac{1}{2}\,\frac{1}{2} = \frac{1}{4} \\ y = \frac{1}{2}\sin(\pi/3) = \frac{1}{2}\,\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\end{array}}$

(b) (5 pts) The polar curve has one positive y-intercept. Find the slope of the tangent line at this point.

$\theta = \pi/2 \qquad r(\pi/2) = 1 - \sin\left(\frac{1}{2}\frac{\pi}{2}\right) = 1 - \frac{\sqrt{2}}{2}$

$\frac{dr}{d\theta} = -\frac{1}{2}\cos\left(\frac{1}{2}\theta\right) \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=\pi/2} = -\frac{1}{2}\cos\left(\frac{1}{2}\frac{\pi}{2}\right) = -\frac{\sqrt{2}}{4}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr/d\theta\,\sin\theta + r\cos\theta}{dr/d\theta\,\cos\theta - r\sin\theta} \overset{\theta=\pi/2}{=} \frac{\left(-\frac{\sqrt{2}}{4}\right)(1) + \left(1-\frac{\sqrt{2}}{2}\right)(0)}{\left(-\frac{\sqrt{2}}{4}\right)(0) - \left(1-\frac{\sqrt{2}}{2}\right)(1)}$$

$$= \boxed{\frac{-\sqrt{2}/4}{-1+\sqrt{2}/2} = \frac{-\sqrt{2}}{-4+2\sqrt{2}}}$$

6 - 2

4. (10 pts) Consider the parametric equations $x = t^2 - 3t$, $y = 12t - t^3$ (curve shown below).

(a) Find the (x, y) coordinates of all locations on the curve at which the tangent line is horizontal.

WANT $\frac{dy}{dx} \overset{?}{=} 0 \iff \frac{dy/dt}{dx/dt} = \frac{12-3t^2}{2t-3} \overset{?}{=} 0$

$12 - 3t^2 = 0 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2$

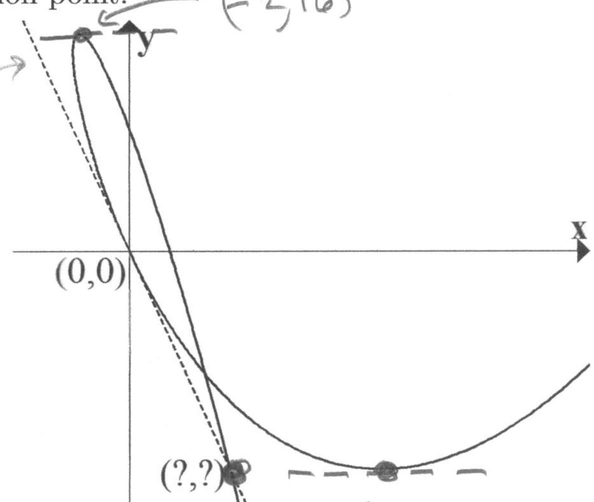
$t = -2 \Rightarrow x = (-2)^2 - 3(-2) = 10,\ y = 12(-2) - (-2)^3 = -16$

$t = 2 \Rightarrow x = (2)^2 - 3(2) = -2,\ y = 12(2) - 2^3 = 16$

$\boxed{(10, -16) \quad \text{AND} \quad (-2, 16)}$

(b) The tangent line at $t = 0$ also intersects the curve at one other point (as shown). Find the (x, y) coordinates of the other intersection point.

$(-2, 16)$



$(10, -16)$

At $t = 0$, $x = 0, y = 0$, and

$\frac{dy}{dx} = \frac{12 - 0}{0 - 3} = -4$

Thus, $y = -4(x - 0) + 0$

$y = -4x$

INTERSECTION:

$12t - t^3 = -4(t^2 - 3t)$

$\cancel{12t} - t^3 = -4t^2 + \cancel{12t}$

$0 = t^3 - 4t^2$

$0 = t^2(t - 4)$ $\begin{cases} t = 0 \text{ or} \\ t = 4 \end{cases}$

$t = 4 \Rightarrow x = (4)^2 - 3(4) = 4,\ y = 12(4) - (4)^3 = -16$

$\boxed{(4, -16)}$

5. (15 pts) At time $t = 0$, an egg is thrown up into the air toward Dr. Loveless by a disgruntled student. The egg's path is described parametrically by $\underbrace{x = t,\ y = 8\sqrt{t+1},\ z = 15t - t^2}_{\vec{r}_1(t)}$.

(a) (3 pts) At the time $t = 0$, find a vector that is tangent to the curve and has length 5.

$\vec{r}_1\,'(t) = \langle 1, \frac{4}{\sqrt{t+1}}, 15-2t \rangle$, $\vec{r}_1\,'(0) = \langle 1, 4, 15 \rangle$

$|\vec{r}_1\,'(0)| = \sqrt{1+16+225} = \sqrt{242} = 11\sqrt{2}$

+ or − okay

$$\boxed{\frac{5}{\sqrt{242}}\langle 1, 4, 15 \rangle = \left\langle \frac{5}{\sqrt{242}}, \frac{20}{\sqrt{242}}, \frac{75}{\sqrt{242}} \right\rangle}$$

(b) Find parametric equations for the tangent line at the positive time when the egg hits the xy-plane.

HITS XY-PLANE $\Leftrightarrow$ $z = 15t - t^2 \stackrel{?}{=} 0 \Leftrightarrow t(15-t) = 0$

$t = 0$ or $t = 15$

$\vec{r}_1(15) = \langle 15, 8\sqrt{16}, 0 \rangle = \langle 15, 32, 0 \rangle$

$\vec{r}_1\,'(15) = \langle 1, \frac{4}{\sqrt{16}}, 15 - 2(15) \rangle = \langle 1, 1, -15 \rangle$

$$\boxed{\begin{aligned} x &= 15 + t \\ y &= 32 + t \\ z &= 0 - 15t \end{aligned}}$$

(c) A rock is also flying through the air following the path $\overbrace{x = 3,\ y = 14 + u,\ z = 28 + u^3}^{\vec{r}_2(u)}$. The path of the rock and the path of the egg intersect (unfortunately for Dr. Loveless, the rock and egg don't collide). Find the (acute) angle of intersection of the two paths. Give your final answer in degrees rounded to two digits after the decimal.

INTERSECTION: $t \stackrel{?}{=} x \stackrel{?}{=} 3$ $\quad t = 3$

$8\sqrt{t+1} \stackrel{?}{=} y \stackrel{?}{=} 14 + u \Rightarrow 16 = 14 + u \Rightarrow u = 2$

$15t - t^2 \stackrel{?}{=} z \stackrel{?}{=} 28 + u^3 \Rightarrow$ CHECK $\underbrace{15(3) - (3)^2}_{36} \stackrel{?}{=} \underbrace{28 + (2)^3}_{36}$ ✓

$\vec{r}_1\,'(3) = \langle 1, 2, 9 \rangle = \vec{u}$

$\vec{r}_2\,'(u) = \langle 0, 1, 3u^2 \rangle$ $\quad \vec{r}_2\,'(2) = \langle 0, 1, 12 \rangle = \vec{v}$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{0 + 2 + 108}{\sqrt{1+4+81}\,\sqrt{0+1+144}} = \frac{110}{\sqrt{86}\sqrt{145}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{110}{\sqrt{86}\sqrt{145}}\right) \approx 9.92 \text{ degrees}}$$