1. (6 pts) You are standing on the surface $z = \frac{x}{y^2+1} - \ln(xy+1) + xye^x$ at the point $(x,y) = (1,1)$. Is it steeper to walk in the positive x-direction or the positive y-direction? (Justify your answer with appropriate partial derivative calculations).

$$\frac{\partial z}{\partial x} = \frac{1}{y^2+1} - \frac{y}{xy+1} + ye^x + xye^x$$

$$\frac{\partial z}{\partial x}(1,1) = \frac{1}{2} - \frac{1}{2} + e^1 + e^1 = \boxed{2e} \leftarrow \text{BIGGER}$$

x-direction steeper

$$\frac{\partial z}{\partial y} = -\frac{2xy}{(y^2+1)^2} - \frac{x}{xy+1} + xe^x$$

$$\frac{\partial z}{\partial y}(1,1) = -\frac{2}{4} - \frac{1}{2} + e^1 = e - 1$$

2. (8 pts) You lose your grip on a balloon in a twister. The balloon's locations is given by the position vector function $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), t + \sin(t) \rangle$. Find the minimum speed AND give the tangential component of acceleration at the first positive time where the minimum speed occurs.

$$\vec{r}\,'(t) = \langle -2\sin(2t),\ 2\cos(2t),\ 1+\cos(t) \rangle$$

$$v(t) = \text{SPEED} = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + (1+\cos(t))^2}$$

$$v(t) = \sqrt{4 + (1+\cos(t))^2}$$

Since $-1 \le \cos(t) \le 1$, $v(t)$ has a minimum of

$$\sqrt{4 + (1-1)^2} = \boxed{2} \leftarrow \text{minimum speed.} \quad (\text{at } t = \pi)$$

Since $a_T = v'(t) = \frac{\cancel{2}(1+\cos(t))(-\sin(t))}{\cancel{2}\sqrt{4+(1+\cos(t))^2}}$

it is zero when speed is at a minimum,

$$\boxed{a_T = 0}$$

3. (10 pts) Find and classify all critical points of the surface $f(x, y) = \frac{1}{6}x^2y + 2x - 5\ln(y) + y$.

(i) $f_x(x,y) = \frac{1}{3}xy + 2 \stackrel{?}{=} 0 \Rightarrow xy = -6$

$y = -\frac{6}{x}$

(ii) $f_y(x,y) = \frac{1}{6}x^2 - \frac{5}{y} + 1 \stackrel{?}{=} 0$

(i) & (ii)

$$\frac{1}{6}x^2 + \frac{5}{6}x + 1 = 0$$

$$x^2 + 5x + 6 = 0$$

$$(x+2)(x+3) = 0$$

$$x = -2, \quad x = -3$$

$$y = \frac{-6}{-2} = 3, \quad y = \frac{-6}{-3} = 2$$

$\boxed{(-2, 3) \qquad (-3, 2)}$

$f_{xx}(x,y) = \frac{1}{3}y, \quad f_{yy}(x,y) = \frac{5}{y^2}, \quad f_{xy}(x,y) = \frac{1}{3}x$

$D(-2,3) = (1)\left(\frac{5}{9}\right) - \left(-\frac{2}{3}\right)^2 = \frac{5}{9} - \frac{4}{9} = 1/9 > 0$

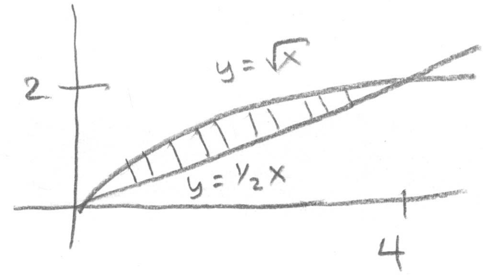
and $f_{xx}(-2,3) > 0$ ∪

$\boxed{(-2,3) \leftarrow \text{LOCAL MIN}}$

$D(-3,2) = \left(\frac{2}{3}\right)\left(\frac{5}{4}\right) - \left(\frac{-3}{3}\right)^2 = \frac{5}{6} - 1 = -1/6 < 0$

$\boxed{(-3,2) \leftarrow \text{SADDLE POINT}}$

4. (a) (8 pts) Evaluate: $\displaystyle\int_0^4 \int_{\frac{1}{2}x}^{\sqrt{x}} \frac{e^y}{y}\, dy dx.$



REVERSE ORDER

$0 \le y \le 2$

$y^2 \le x \le 2y$

$\int_0^2 \int_{y^2}^{2y} \frac{e^y}{y}\, dx dy = \int_0^2 2e^y - ye^y\, dy$

$= 2e^y\Big|_0^2 - \int_0^2 ye^y dy$ $\qquad u=y \quad dv=e^y dy$, $du=dy \quad v=e^y$

$= 2e^2 - 2 - \left[ye^y\Big|_0^2 - \int_0^2 e^y dy\right]$

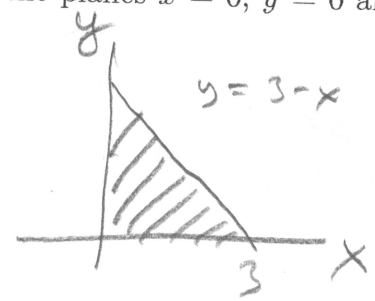
$= 2e^2 - 2 - [(2e^2-0) - (e^2-1)]$

$= 2e^2 - 2 - [e^2+1]$

$= \boxed{e^2 - 3}$

(b) (8 pts) Set up and evaluate a double integral to find the volume of the solid that is bounded below by $z = 0$, bounded above by $z = x^2$, and bounded by the planes $x = 0$, $y = 0$ and $x + y = 3$.

NOTE
- $z=x^2$
$z=0$
$\subset$ y-axis



$\int_0^3 \int_0^{3-x} x^2\, dy dx$

$\int_0^3 x^2(3-x)dx$

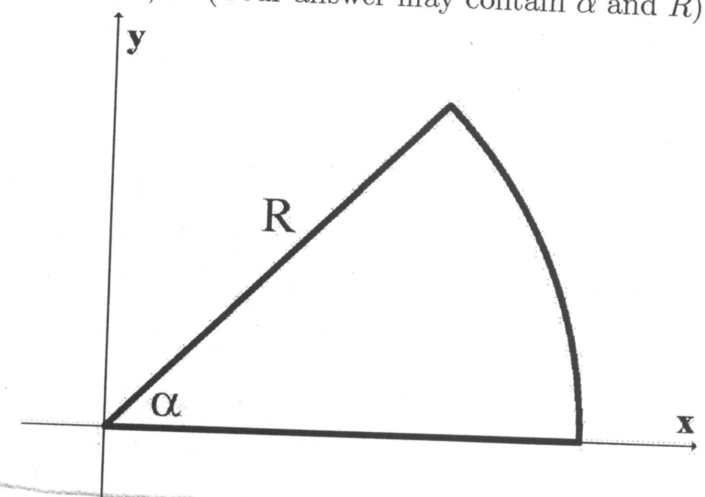
$= \int_0^3 3x^2 - x^3 dx = x^3 - \frac{1}{4}x^4\Big|_0^3$

$= (3)^3 - \frac{1}{4}(3)^4 = 3^3(1-3/4) = \boxed{\frac{27}{4} = 6.75}$

5. Dr. Loveless likes flat circular wedges (especial if they come with pie on top). Consider a thin wedge shaped lamina with angle α $(0 \le \alpha \le 2\pi)$, radius R and vertex at the origin as shown. Assume the density of the wedge is proportional to the distance from the origin.

(a) (8 pts) Find the x-coordinate of the center of mass, $\bar{x}$. (Your answer may contain α and R).

$$\bar{x} = \frac{\iint_D x \cdot k\sqrt{x^2+y^2}\,dA}{\iint_D k\sqrt{x^2+y^2}\,dA} = \frac{M_y}{M}$$



• POLAR $\quad 0 \le \theta \le \alpha$

$\quad\quad\quad\quad 0 \le r \le R$

$$M = \int_0^\alpha \int_0^R kr\, r\,dr\,d\theta$$
$$= k\int_0^\alpha d\theta \int_0^R r^2\,dr$$
$$= k\alpha \tfrac{1}{3} r^3\Big|_0^R = \tfrac{1}{3}k\alpha R^3 = \text{TOTAL MASS}$$

$$M_y = \int_0^\alpha \int_0^R r\cos\theta\, kr\, r\,dr\,d\theta = k\int_0^\alpha \cos\theta\,d\theta \int_0^R r^3\,dr$$
$$= k\sin(\alpha)\tfrac{1}{4}R^4 = \tfrac{1}{4}k\sin(\alpha)R^4 = M_y$$

$$\bar{X} = \frac{M_y}{M} = \frac{\frac{1}{4}k\sin(\alpha)R^4}{\frac{1}{3}k\alpha R^3} = \boxed{\frac{3}{4}\frac{\sin\alpha}{\alpha}R}$$

(b) (2 pts) What point (x, y) does the center of mass approach as α goes to zero? (Your answer may contain R).

$\bar{y}$ will approach zero

$\bar{x}$ will approach $\lim_{\alpha\to 0} \frac{3}{4}\frac{\sin\alpha}{\alpha}R = \frac{3}{4}R$

$$\boxed{\left(\tfrac{3}{4}R, 0\right)}$$