1. (11 pts) **The two parts below are not related. Give answers in simplified exact form.**

(a) Consider the position vector function $\mathbf{r}(t) = \langle 5t, e^t, e^{-3t} \rangle$. Find all values of t at which the tangential component of acceleration is zero.

$$a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|} = \frac{\langle 5, e^t, -3e^{-3t} \rangle \cdot \langle 0, e^t, 9e^{-3t} \rangle}{|\vec{r}\,'|} \stackrel{?}{=} 0$$

$$\Rightarrow 0 + e^{2t} - 27e^{-6t} \stackrel{?}{=} 0$$

$$e^{2t} - \frac{27}{e^{6t}} \stackrel{?}{=} 0$$

$$e^{8t} - 27 \stackrel{?}{=} 0$$

$$e^{8t} = 27$$

$$8t = \ln(27)$$

$$\boxed{t = \frac{\ln(27)}{8}} \approx 0.41198$$

(b) Find the tangent plane for $z = f(x,y) = x^5 \sin\left(\frac{\pi x}{y^2}\right) + \ln(y) + 4$ at $(x,y) = (1,1)$.

$$f(1,1) = \sin(\pi) + \ln(1) + 4 = 4$$

$$f_x = 5x^4 \sin\left(\frac{\pi}{y^2}x\right) + \frac{\pi}{y^2} x^5 \cos\left(\frac{\pi}{y^2}x\right)$$

$$\Rightarrow f_x(1,1) = 5\sin(\pi) + \pi\cos(\pi) = -\pi$$

$$f_y = -2y^{-3}\pi x^6 \cos\left(\frac{\pi x}{y^2}\right) + \frac{1}{y}$$

$$\Rightarrow f_y(1,1) = -2\pi\cos(\pi) + 1 = 2\pi + 1$$

$$\boxed{z - 4 = -\pi(x-1) + (2\pi+1)(y-1)}$$

$$z - 4 = -\pi x + \pi + (2\pi+1)y - 2\pi - 1$$

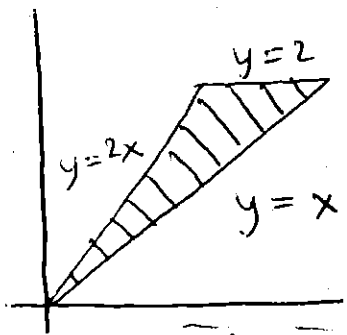
$$z = -\pi x + (2\pi+1)y + 3 - \pi$$

2. (14 pts) **The two parts below are not related.**

(a) Find the volume under the plane $8x + 2y - z = 0$ and above the region enclosed by $y = x$, $y = 2x$, and $y = 2$.

$$\iint_D 8x + 2y \, dA = \int_0^2 \left(\int_{\frac{y}{2}}^{y} 8x + 2y \, dx \right) dy$$

$$= \int_0^2 4x^2 + 2yx \Big|_{y/2}^{y} dy$$



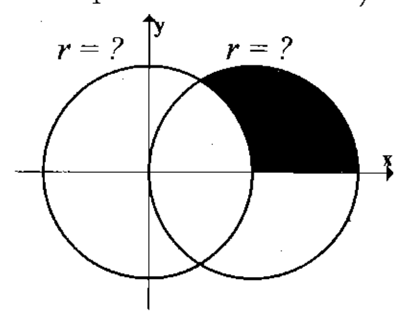
$$= \int_0^2 (4y^2 + 2y^2) - (y^2 + y^2) \, dy$$

$$= \int_0^2 4y^2 \, dy$$

$$= \frac{4}{3} y^3 \Big|_0^2 = \boxed{\frac{32}{3}} \approx 10.\overline{6}$$

OR $\displaystyle \int_0^1 \int_x^{2x} 8x + 2y \, dy\, dx + \int_1^2 \int_x^{2} 8x + 2y \, dy\, dx = \frac{11}{3} + 7 = \frac{32}{3}$

(b) Evaluate $\iint_D 3\, dA$, where D is the region shown which is inside the circle $x^2 + y^2 = 4x$, outside $x^2 + y^2 = 4$ and in the first quadrant. (Hint: Convert to find polar functions first).



NOTE: $x^2 + y^2 = 4 \Rightarrow r = 2$

$x^2 + y^2 = 4x \Rightarrow r^2 = 4r\cos\theta$

$r = 4\cos\theta$

INTERSECTION: $4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \pi/3$

$$\int_0^{\pi/3} \int_2^{4\cos\theta} 3r \, dr \, d\theta$$

$$= \int_0^{\pi/3} \frac{3}{2} r^2 \Big|_2^{4\cos\theta} d\theta$$

$$= \int_0^{\pi/3} \frac{3}{2} 16\cos^2\theta - \frac{3}{2} 4 \, d\theta$$

$$= \int_0^{\pi/3} 24 \tfrac{1}{2}(1 + \cos(2\theta)) - 6 \, d\theta$$

$$= \int_0^{\pi/3} 12\cos(2\theta) + 6 \, d\theta$$

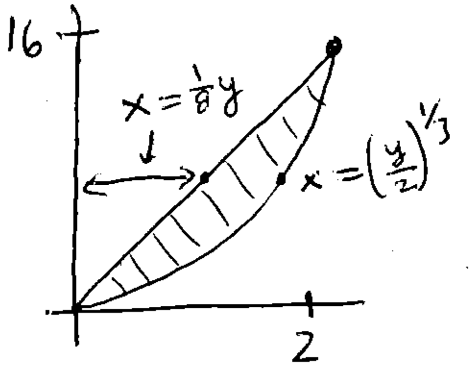
$\sqrt{3}/2$

$$= 6\sin(2\theta) + 6\theta \Big|_0^{\pi/3} = 6\sin\left(\tfrac{2\pi}{3}\right) + 2\pi$$

$$= \boxed{3\sqrt{3} + 2\pi} \approx 11.4793$$

3. (13 pts) **The two parts below are not related.**

(a) Draw the region of integration and reverse the order of integration for $\int_0^2 \int_{2x^3}^{8x} g(x,y)\,dy\,dx$. (Do NOT evaluate, just rewrite in the opposite order $dx\,dy$).



$$\int_0^{16} \left(\int_{\frac{1}{8}y}^{\left(\frac{y}{2}\right)^{1/3}} g(x,y)\,dx \right) dy$$

(b) Find the absolute (global) max and min of $f(x,y) = 4x^2 - xy + 9$ over the triangular region with corners at $(0,0)$, $(0,4)$ and $(2,4)$. For full credit, you MUST clearly find the critical point(s) and show appropriate work for every boundary.

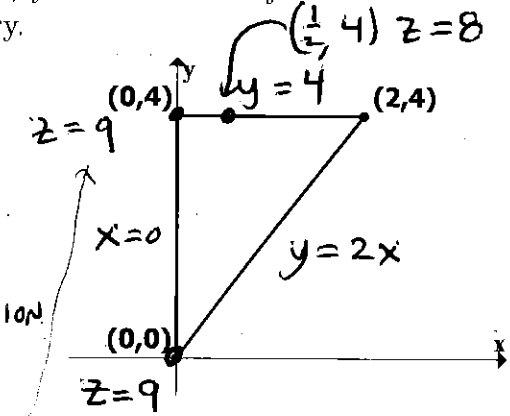
INSIDE CRITICAL PTS?

$f_x = 8x - y \overset{?}{=} 0 \qquad y = 0$

$f_y = -x \overset{?}{=} 0 \rightarrow x = 0$

$(0,0)$ ON BOUNDARY

NO CRITICAL PTS PROPERLY INSIDE REGION



BOUNDARY

I) $x = 0 \Rightarrow z = 9 \leftarrow$ CONSTANT

II) $y = 4 \Rightarrow z = 4x^2 - 4x + 9$

$z' = 8x - 4 \overset{?}{=} 0 \Rightarrow x = \frac{1}{2} \Rightarrow z = 4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 9$

$= 1 - 2 + 9$

$= 8$

III) $y = 2x \Rightarrow z = 4x^2 - 2x^2 + 9 = 2x^2 + 9$

$z' = 4x \overset{?}{=} 0 \Rightarrow x = 0$

MUST BE AT ONE OF

$(0,0) \longrightarrow z = 9$

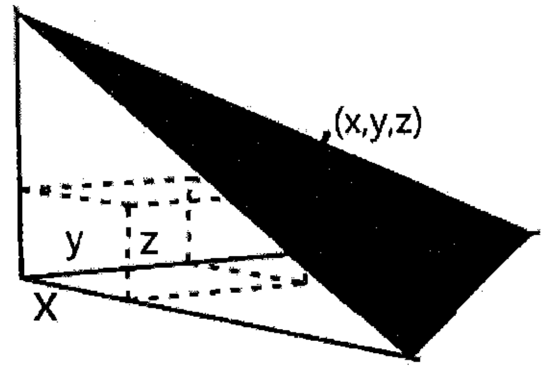
$(0,4) \longrightarrow z = 9$

$(\frac{1}{2},4) \longrightarrow \boxed{z = 8} \leftarrow$ GLOBAL MIN

$(2,4) \longrightarrow z = 16 - 8 + 9 = \boxed{17} \leftarrow$ GLOBAL MAX

4. (12 pts) Find the x, y, z dimensions of the rectangular box with maximum volume in the first octant with all vertices (corners) in the coordinate planes except one vertex (corner) that is on the plane $4x + 3y + z = 12$. (One example of such a rectangular box is shown)

At the end, ***clearly*** use the 2nd derivative test to verify your point gives a local max.



MAXIMIZE

VOLUME OF BOX $= xyz$

SUBJECT TO CONSTRAINT

$z = 12 - 4x - 3y$

$\text{VOL} = f(x,y) = 12xy - 4x^2y - 3xy^2$

$f_x = 12y - 8xy - 3y^2 \overset{?}{=} 0 \Rightarrow y(12 - 8x - 3y) \overset{?}{=} 0$

$f_y = 12x - 4x^2 - 6xy \overset{?}{=} 0 \quad 2x(6 - 2x - 3y) \overset{?}{=} 0$

$12 - 8x - 3y \overset{?}{=} 0 \Rightarrow y = \frac{12 - 8x}{3}$

COMBINE $\quad 6 - 2x - 3\left(\frac{12-8x}{3}\right) \overset{?}{=} 0$

$6 - 2x - 12 + 8x \overset{?}{=} 0 \Rightarrow 6x = 6 \Rightarrow \boxed{x = 1}$

$\boxed{y = 4/3}$

$\boxed{z = 12 - 4 - 4 = 4}$

$\boxed{(1, \frac{4}{3}, 4) = (x, y, z)}$

2ND DERIV. TEST

$f_{xx} = -8y \qquad f_{xx}(1, 4/3) = -\frac{32}{3} < 0$

$f_{yy} = -6x \qquad f_{yy}(1, 4/3) = -6 < 0$

$f_{xy} = 12 - 8x - 6y \qquad f_{xy}(1, 4/3) = 12 - 8 - 8 = -4$

$D = \left(-\frac{32}{3}\right)(-6) - (-4)^2 = 64 - 16 = 48 > 0$

} LOCAL MAX!!!