1. (13 pts)

(a) Find all values of t at which the tangential and normal components of acceleration are equal for the curve $\mathbf{r}(t) = \langle 2t, 1-3t, t^2 \rangle$.

$\vec{r}'(t) = \langle 2, -3, 2t \rangle$ $\quad \vec{r}''(t) = \langle 0, 0, 2 \rangle$

$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 2t \\ 0 & 0 & 2 \end{vmatrix} = (-6-0)\vec{i} - (4-0)\vec{j} + (0-0)\vec{k}$

$= \langle -6, -4, 0 \rangle$

$a_T \overset{?}{=} a_N \Rightarrow \dfrac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|} \overset{?}{=} \dfrac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|}$

$0 + 0 + 4t = \sqrt{36 + 16 + 0}$

$\boxed{t = \frac{\sqrt{52}}{4}}$ ≈ 1.80278

also can be written as

$\frac{1}{2}\sqrt{13} = \frac{13}{2\sqrt{13}} = \frac{13}{\sqrt{52}}$

(b) The surface area of a solid circular cylinder of radius r and height h is $S(r, h) = 2\pi r^2 + 2\pi rh$.

i. Give the total differential for the surface area.

$dS = (4\pi r + 2\pi h)dr + (2\pi r)dh$

ii. Estimate the surface area when $r = 2.1$ inches and $h = 9.8$ inches using the linear approximation at $(r, h) = (2, 10)$.

at $(2, 10)$ $\quad dS = (4\pi(2) + 2\pi(10))dr + (2\pi(2))\,dh$

$dS = 28\pi\, dr + 4\pi\, dh$

$dr = 0.1,\ dh = -0.2$

$dS = 2.8\pi - 0.8\pi = 2\pi$

$S(2,10) = 2\pi(2)^2 + 2\pi(2)(10) = 8\pi + 40\pi = 48\pi$

$48\pi + 2\pi = \boxed{50\pi}$

2. (14 pts)

(a) Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ for $z^3 - 8z - e^2 = x^2\sqrt{y} + \ln(y) - e^{xy^3} \cancel{+1}\ -2$ at the point $(x, y, z) = (2, 1, 3)$.

$$3z^2\frac{\partial z}{\partial x} - 8\frac{\partial z}{\partial x} = 2x\sqrt{y} - y^3e^{xy^3}$$

$$\frac{\partial z}{\partial x} = \frac{2x\sqrt{y} - y^3e^{xy^3}}{3z^2 - 8}$$

$$\frac{\partial z}{\partial x} = \frac{2(2)\sqrt{1} - 1^3e^2}{3(3)^2 - 8} = \boxed{\frac{4 - e^2}{19}} \approx -0.17837$$

(b) Consider $\displaystyle\int_0^8 \int_{y/4}^{\sqrt[3]{y}} 5x\, dx dy$. Complete the following two tasks:

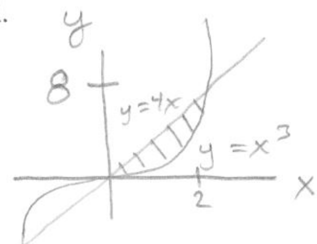
- Reverse the order of integration to give the equivalent integral in the order $dydx$.
- Using either order, compute the value of this double integral.

$0 \le y \le 8$

$\frac{1}{4}y \le x \le \sqrt[3]{y}$

$x = \sqrt[3]{y} \Rightarrow y = x^3$

$x = \frac{1}{4}y \Rightarrow y = 4x$



$$\int_0^2 \int_{x^3}^{4x} 5x\, dy\, dx$$

$$= \int_0^2 5xy\Big|_{x^3}^{4x} dx$$

$$= \int_0^2 20x^2 - 5x^4 dx$$

$$= \frac{20}{3}x^3 - \frac{5}{5}x^5\Big|_0^2$$

$$= \frac{20}{3}2^3 - 2^5$$

$$= \frac{160}{3} - 32 = \frac{160 - 96}{3}$$

$$= \boxed{\frac{64}{3}} = 21.\bar{3}$$

OR

$$\int_0^8 \int_{y/4}^{\sqrt[3]{y}} 5x\, dx\, dy$$

$$= \int_0^8 \frac{5}{2}x^2\Big|_{y/4}^{\sqrt[3]{y}} dy$$

$$= \frac{5}{2}\int_0^8 y^{2/3} - \frac{1}{16}y^2 dy$$

$$= \frac{5}{2}\left(\frac{3}{5}y^{5/3} - \frac{1}{48}y^3\Big|_0^8\right)$$

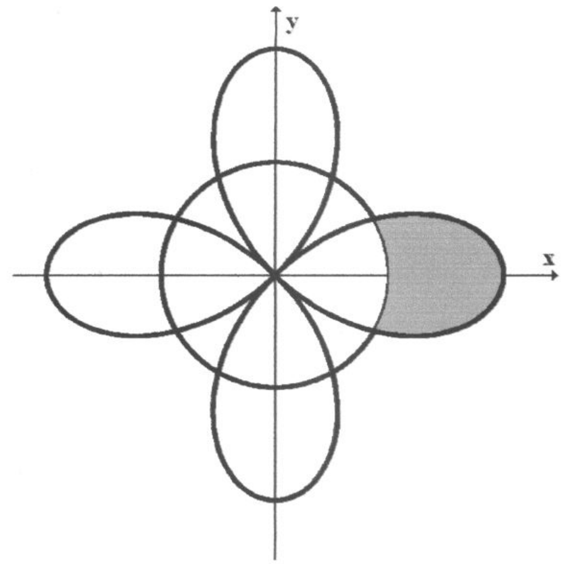
$$= \frac{3}{2}8^{5/3} - \frac{5}{96}8^{3}$$

$$= 48 - \frac{320}{12}$$

$$= 48 - \frac{80}{3}$$

$$= \frac{144 - 80}{3} = \boxed{\frac{64}{3}} = 21.\bar{3}$$

3. (10 pts) Find the area of the region outside the circle $x^2 + y^2 = \frac{1}{4}$ and inside one loop of the polar curve $r = \cos(2\theta)$.



INSIDE BOUND: $x^2+y^2=\frac{1}{4} \Leftrightarrow r=\frac{1}{2}$

OUTSIDE BOUND: $r=\cos(2\theta)$

INTERSECT:

$\cos(2\theta)=\frac{1}{2}$

$\Leftrightarrow 2\theta = \dots, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \dots$ $\quad +2\pi$ $\quad +2\pi$

$\Leftrightarrow \theta = \dots, \underbrace{-\frac{\pi}{6}, \frac{\pi}{6}}_{\text{THESE TWO!}}, \frac{5\pi}{6}, \frac{7\pi}{6}, \dots$

BOUNDS

$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$

$\frac{1}{2} \le r \le \cos(2\theta)$

$$\text{AREA} = \iint_D 1\,dA = \int_{-\pi/6}^{\pi/6}\int_{1/2}^{\cos(2\theta)} 1\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2}r^2\Big|_{1/2}^{\cos(2\theta)}\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2(2\theta) - \frac{1}{2}\left(\frac{1}{2}\right)^2 d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\frac{1}{2}(1+\cos(4\theta)) - \frac{1}{8}\,d\theta \qquad \frac{1}{4}-\frac{1}{8}=\frac{1}{8}$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{8} + \frac{1}{4}\cos(4\theta)\,d\theta$$

$$= \frac{1}{8}\theta + \frac{1}{16}\sin(4\theta)\Big|_{-\pi/6}^{\pi/6}$$

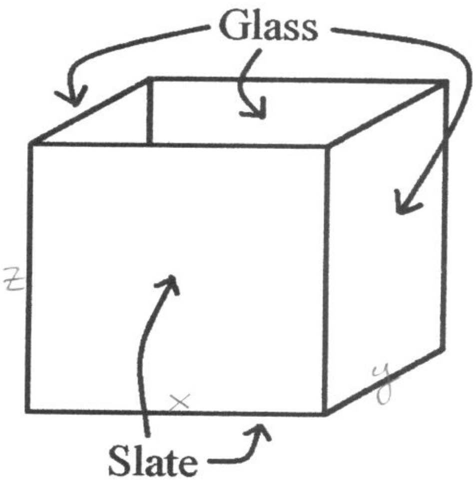
$$= \left(\frac{\pi}{48} + \frac{1}{16}\sin\left(\frac{4\pi}{6}\right)\right) - \left(-\frac{\pi}{48} + \frac{1}{16}\sin\left(-\frac{4\pi}{6}\right)\right)$$

$$= \frac{2\pi}{48} + \frac{1}{16}\frac{\sqrt{3}}{2} - \frac{1}{16}\frac{-\sqrt{3}}{2}$$

$$= \boxed{\frac{\pi}{24} + \frac{\sqrt{3}}{16}}$$

4. (13 pts) You are designing a box that does not have a top (as shown).
The volume must be 10 cubic feet.
The bottom and front are are both going to be made out of slate that costs $5.00 per square foot.
The other three sides are made of glass that costs $2.00 per square foot.
Find the minimum cost.

Verify that your critical point is a local minimum by using the second derivative test.
(To speed up your work, you do not have to give exact answers here. Instead you should give decimal values correct to two digits after the decimal).



$\boxed{\text{OBJECTIVE}}$

$$\text{COST} = 5(xy + xz) + 2(2yz + xz)$$
$$= 5xy + 7xz + 4yz$$

$\boxed{\text{CONSTRAINT}}$

$$xyz = 10 \Rightarrow z = \frac{10}{xy}$$

Thus,

$$C(x,y) = 5xy + \frac{70}{y} + \frac{40}{x}$$

$\boxed{\text{CRITICAL PTS}}$

(i) $C_x = 5y - \frac{40}{x^2} \overset{?}{=} 0 \Rightarrow y = \frac{8}{x^2}$

(ii) $C_y = 5x - \frac{70}{y^2} \overset{?}{=} 0 \Rightarrow xy^2 = 14$

combine (i) & (ii) $\Rightarrow x\left(\frac{8}{x^2}\right)^2 = 14 \Rightarrow \frac{64}{x^3} = 14 \Rightarrow x^3 = \frac{64}{14} = \frac{32}{7}$

So $x = \left(\frac{32}{7}\right)^{1/3} \approx 1.659653067$

$y = \frac{8}{x^2} \approx 2.904392867$

EXACT $= 30\sqrt[3]{14}$?

$$C(1.65965, 2.90439) = 5(1.65965)(2.90439) + \frac{70}{2.90439} + \frac{40}{1.65965} \approx 72.3042679$$

MINIMUM COST $\longrightarrow$ $\boxed{\$72.30}$

$\boxed{2^{\text{nd}} \text{ DERIV. TEST}}$

$$C_{xx} = \frac{80}{x^3},\quad C_{yy} = \frac{140}{y^3},\quad C_{xy} = 5,\quad D = C_{xx}C_{yy} - C_{xy}^2$$

at the critical pt. $C_{xx} = 17.5$, $C_{yy} \approx 5.714286$, $D = 75$

$D > 0$ AND $C_{xx} > 0$ (AND $C_{yy} > 0$)

So the critical point is a local minimum.