Math 126, Section E, Spring 2009, Solutions to Midterm I

1. Find the line of intersection of the two planes $x - 3y + z = 9$ and $-x + 4y = 4$. Give your answer

   (a) As a vector function. The direction vector for the line is going to be perpendicular to the normal vectors of both planes so

   $$\mathbf{v} =< 1, -3, 1 > \times < -1, 4, 0 >=< -4, -1, 1 > .$$

   To find a point common to both planes we can set $x = 0$ which gives $-3y + z = 9$ and $4y = 4$ so a point will be $(0, 1, 12)$. The vector equation of the line is

   $$\mathbf{r}(t) =< 0 - 4t, 1 - t, 12 + t > .$$

   (b) As a parametric curve.

   $$x = 0 - 4t, \qquad y = 1 - t, \qquad z = 12 + t.$$

   (c) With symmetric equations.

   $$-\frac{x}{4} = 1 - y = z - 12$$

2. Find the angle of intersection of the two curves $\mathbf{r}_1(t) =< t^3, 2t^2 + 1, 2t + 3 >$ and $\mathbf{r}_2(s) =< s - 4, s - 3, s - 1 >$.

   The angle of intersection of two curves is the angle between their tangent vectors at that point. So first we need to see where (if) they intersect.

   $$< t^3, 2t^2 + 1, 2t + 3 >=< s - 4, s - 3, s - 1 >$$

   give $t = 0$ and $s = 4$. So they intersect at the point $\mathbf{r}_1(0) =< 0, 1, 3 >= \mathbf{r}_2(4)$. Their tangent vectors are given by the values of the derivatives:

   $$\mathbf{r}_1'(t) =< 3t^2, 4t, 2 > \qquad \text{and} \qquad \mathbf{r}_1'(0) =< 0, 0, 2 >$$

   and

   $$\mathbf{r}_2'(s) =< 1, 1, 1 > \qquad \text{and} \qquad \mathbf{r}_2'(4) =< 1, 1, 1 >$$

   so the angle between them can be calculated from

   $$\cos\theta = \frac{< 0, 0, 2 > \cdot < 1, 1, 1 >}{|| < 0, 0, 2 > | < 1, 1, 1 > |} = \frac{1}{\sqrt{3}}$$

3. Given the points $A(1,2,3)$, $B(0,0,5)$, $C(2,3,0)$ and $D(2,0,1)$:

(a) Find the equation of the plane containing the three points A, B, and C. Hint: Check you answer to see A, B and C are on your plane!
A normal of the plane can be calculated in many ways using the cross product. For example,

$$\mathbf{n} = \vec{BA} \times \vec{BC} = < 1, 2, -2 > \times < 2, 3, -5 > = < -4, 1, -1 >$$

Any one of the three points will then give

$$-4x + y - z = -5$$

(b) What is the area of the triangle ABC?
It is $\frac{1}{2}| < -4, 1, -1 > | = \frac{3\sqrt{2}}{2}$

(c) Find the distance from point D to the plane in part (a)
The distance from D to the plane can be computed as

$$|\text{comp}_{\mathbf{n}} \vec{BD}| = \frac{|\mathbf{n} \cdot \vec{BD}|}{\mathbf{n} \cdot \mathbf{n}} = \frac{| < -4, 1, -1 > \cdot < 2, 0, -4 > |}{< -4, 1, -1 > \cdot < -4, 1, -1 >} = \frac{2}{9}$$

(d) If you draw a perpendicular line from point D to the plane, where does it intersect the plane?
The line through D perpendicular to the plane is

$$\mathbf{r}(t) = < 2 - 4t, 0 + t, 1 - t >$$

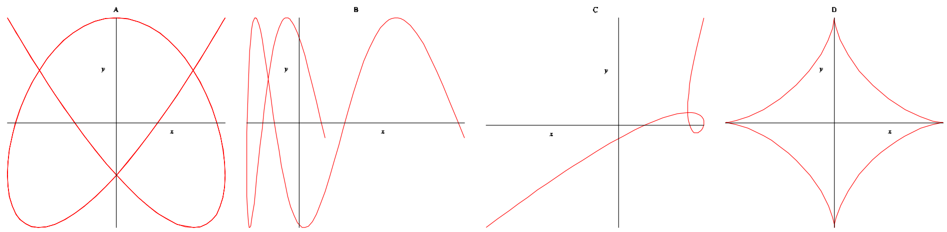
which intersects the plane when

$$-4(2 - 4t) + (t) - (1 - t) = -5$$

which is when $t = 2/9$ so the point is $\mathbf{r}(2/9) = < 10/9, 2/9, 7/9 >$

4. (a) Match the following parametric curves with their graphs.

   1. $x = \sin^3 t, y = \cos^3 t$ D
   2. $x = t^2 - 4t - 20, y = \cos t$ B
   3. $x = \sin(3t), y = \cos(4t)$ A
   4. $x = t^3 - 4t^2 + 50, y = t^3 - 5t + 1$ C



(b) Find the equation of the tangent line to $\mathbf{r}(t) =< \sin(3t), \cos(4t) >$ at the point $(\frac{\sqrt{2}}{2}, -1)$. The slope of the tangent is given by the value of dy/dx at that point.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4\sin(4t)}{3\cos(3t)}.$$

The point corresponds to $t = \pi/4$ so the slope is 0. Therefore, the equation of the tangent line is $y = -1$.

(c) Determine if is concave up or concave down at the point $(\frac{\sqrt{2}}{2}, -1)$. Show your work. Use the appropriate graph above to verify your answer, not to find it!

Concavity is determined by the value of

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{dx/dt} = \frac{\frac{d}{dt}\frac{-4\sin(4t)}{3\cos(3t)}}{3\cos(3t)} = \frac{-16\cos(4t)\cos(3t) - 12\sin(4t)\sin(3t)}{9\cos^3(3t}.$$

When $t = \pi/4$ the values is $8/9$ which is positive so the curve is concave up at $(\frac{\sqrt{2}}{2}, -1)$.