

1. (11 pts) The two parts below are not related. Give answers in simplified exact form.

(a) Consider the position vector function $\mathbf{r}(t) = \langle 5t, e^t, e^{-3t} \rangle$. Find all values of t at which the tangential component of acceleration is zero.

$$a_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|} = \frac{\langle 5, e^t, -3e^{-3t} \rangle \cdot \langle 0, e^t, -9e^{-3t} \rangle}{|\mathbf{r}'|} \stackrel{?}{=} 0$$

$$\Rightarrow 0 + e^{2t} - 27e^{-6t} \stackrel{?}{=} 0$$

$$e^{2t} - \frac{27}{e^{6t}} \stackrel{?}{=} 0$$

$$e^{8t} - 27 \stackrel{?}{=} 0$$

$$e^{8t} = 27$$

$$8t = \ln(27)$$

$$\boxed{t = \frac{\ln(27)}{8}}$$

$$\approx 0.41198$$

(b) Find the tangent plane for $z = f(x, y) = x^5 \sin\left(\frac{\pi x}{y^2}\right) + \ln(y) + 4$ at $(x, y) = (1, 1)$.

$$f(1, 1) = \sin(\pi) + \ln(1) + 4 = 4$$

$$f_x = 5x^4 \sin\left(\frac{\pi x}{y^2}\right) + \frac{\pi}{y^2} x^5 \cos\left(\frac{\pi x}{y^2}\right)$$

$$\Rightarrow f_x(1, 1) = 5 \sin(\pi) + \pi \cos(\pi) = -\pi$$

$$f_y = -2xy^{-3} \pi x^5 \cos\left(\frac{\pi x}{y^2}\right) + \frac{1}{y}$$

$$\Rightarrow f_y(1, 1) = -2(1) \cos(\pi) + 1 = 2\pi + 1$$

$$\boxed{z - 4 = -\pi(x - 1) + (2\pi + 1)(y - 1)}$$

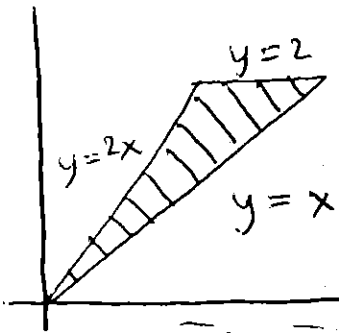
$$z - 4 = -\pi x + \pi + (2\pi + 1)y - 2\pi - 1$$

$$z = -\pi x + (2\pi + 1)y + 3 - \pi$$

2. (14 pts) The two parts below are not related.

- (a) Find the volume under the plane $8x + 2y - z = 0$ and above the region enclosed by $y = x$, $y = 2x$, and $y = 2$.

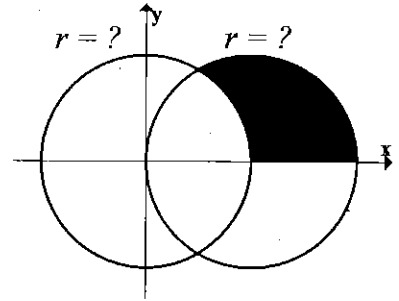
$$\begin{aligned} \iint_D 8x + 2y \, dA &= \int_0^2 \left(\int_{y/2}^y 8x + 2y \, dx \right) dy \\ &= \int_0^2 \left(4x^2 + 2yx \Big|_{y/2}^y \right) dy \\ &= \int_0^2 (4y^2 + 2y^2) - (y^2 + y^2) \, dy \\ &= \int_0^2 4y^2 \, dy \\ &= \frac{4}{3} y^3 \Big|_0^2 = \boxed{\frac{32}{3}} \approx 10.6 \end{aligned}$$



OR $\int_0^1 \int_x^{2x} 8x + 2y \, dy \, dx + \int_1^2 \int_x^2 8x + 2y \, dy \, dx = \frac{11}{3} + 7 = \frac{32}{3}$

- (b) Evaluate $\iint_D 3 \, dA$, where D is the region shown which is inside the circle $x^2 + y^2 = 4x$, outside $x^2 + y^2 = 4$ and in the first quadrant. (Hint: Convert to find polar functions first).

NOTE: $x^2 + y^2 = 4 \Rightarrow r = 2$
 $x^2 + y^2 = 4x \Rightarrow r^2 = 4r \cos \theta$
 $r = 4 \cos \theta$

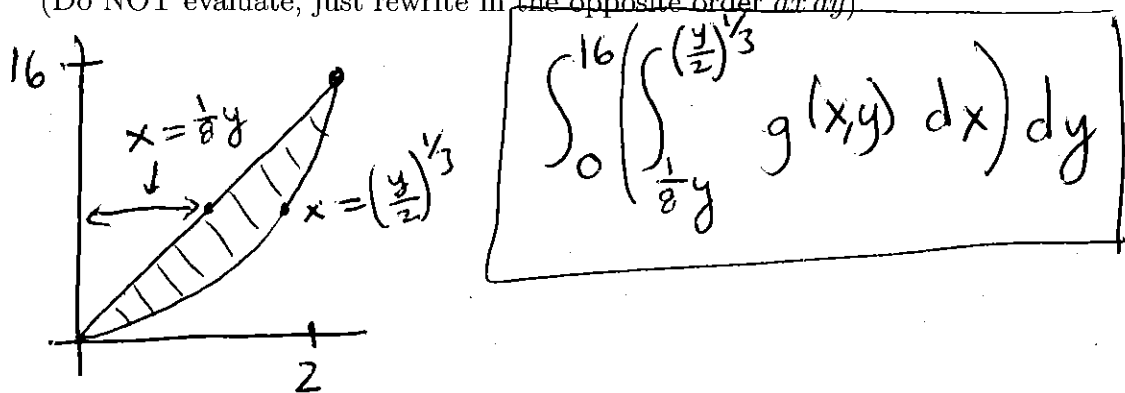


INTERSECTION: $4 \cos \theta = 2 \Rightarrow \cos \theta = \frac{1}{2}$
 $\theta = \pi/3$

$$\begin{aligned} &\int_0^{\pi/3} \int_2^{4 \cos \theta} 3r \, dr \, d\theta \\ &= \int_0^{\pi/3} \frac{3}{2} r^2 \Big|_2^{4 \cos \theta} \, d\theta \\ &= \int_0^{\pi/3} \frac{3}{2} (16 \cos^2 \theta - 4) \, d\theta \\ &= \int_0^{\pi/3} 24 \frac{1}{2} (1 + \cos(2\theta)) - 6 \, d\theta \\ &= \int_0^{\pi/3} 12 \cos(2\theta) + 6 \, d\theta \\ &= 6 \sin(2\theta) + 6\theta \Big|_0^{\pi/3} = 6 \sin\left(\frac{2\pi}{3}\right) + 2\pi \\ &= \boxed{3\sqrt{3} + 2\pi} \approx 11.4793 \end{aligned}$$

3. (13 pts) The two parts below are not related.

- (a) Draw the region of integration and reverse the order of integration for $\int_0^2 \int_{2x^3}^{8x} g(x,y) dy dx$.
 (Do NOT evaluate, just rewrite in the opposite order $dx dy$)



- (b) Find the absolute (global) max and min of $f(x,y) = 4x^2 - xy + 9$ over the triangular region with corners at $(0,0)$, $(0,4)$ and $(2,4)$. For full credit, you MUST clearly find the critical point(s) and show appropriate work for every boundary.

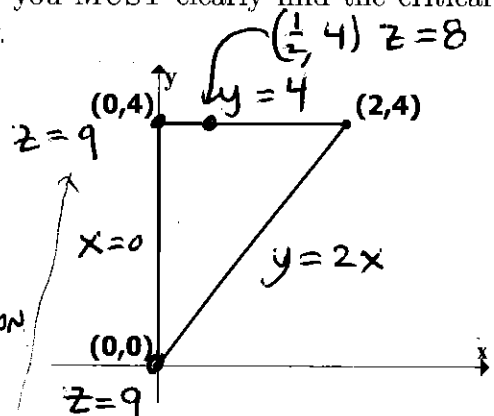
INSIDE CRITICAL PTS?

$$f_x = 8x - y \stackrel{?}{=} 0 \quad \leftarrow y=0$$

$$f_y = -x \stackrel{?}{=} 0 \rightarrow x=0$$

$(0,0)$ ON BOUNDARY

NO CRITICAL PTS PROPERLY INSIDE REGION



BOUNDARY

I) $x=0 \Rightarrow z=9 \leftarrow$ CONSTANT

II) $y=4 \Rightarrow z = 4x^2 - 4x + 9$
 $z' = 8x - 4 \stackrel{?}{=} 0 \Rightarrow x = \frac{1}{2} \Rightarrow z = 4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 9 = 1 - 2 + 9 = 8$

III) $y=2x \Rightarrow z = 4x^2 - 2x^2 + 9 = 2x^2 + 9$
 $z' = 4x \stackrel{?}{=} 0 \Rightarrow x=0$

MUST BE AT

ONE OF

$(0,0)$	\rightarrow	$z=9$	
$(0,4)$	\rightarrow	$z=9$	
$(\frac{1}{2}, 4)$	\rightarrow	$z=8$	\leftarrow GLOBAL MIN
$(2,4)$	\rightarrow	$z = 16 - 8 + 9 = 17$	\leftarrow GLOBAL MAX

4. (12 pts) Find the x, y, z dimensions of the rectangular box with maximum volume in the first octant with all vertices (corners) in the coordinate planes except one vertex (corner) that is on the plane $4x + 3y + z = 12$. (One example of such a rectangular box is shown)

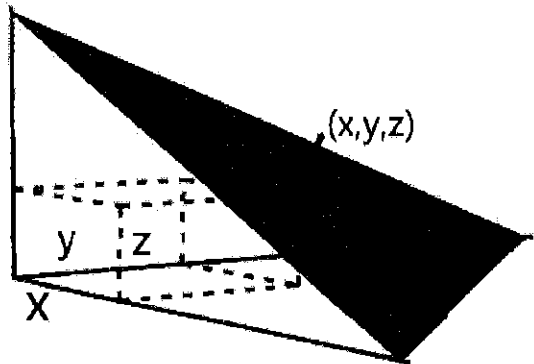
At the end, *clearly* use the 2nd derivative test to verify your point gives a local max.

MAXIMIZE

$$\text{VOLUME OF BOX} = xyz$$

SUBJECT TO CONSTRAINT

$$z = 12 - 4x - 3y$$



$$\text{VOL} = f(x, y) = 12xy - 4x^2y - 3xy^2$$

$$f_x = 12y - 8xy - 3y^2 \stackrel{?}{=} 0 \Rightarrow y(12 - 8x - 3y) \stackrel{?}{=} 0$$

$$f_y = 12x - 4x^2 - 6xy \stackrel{?}{=} 0 \Rightarrow 2x(6 - 2x - 3y) \stackrel{?}{=} 0$$

$$12 - 8x - 3y \stackrel{?}{=} 0 \Rightarrow y = \frac{12 - 8x}{3}$$

COMBINE $6 - 2x - 3\left(\frac{12 - 8x}{3}\right) \stackrel{?}{=} 0$

$$6 - 2x - 12 + 8x \stackrel{?}{=} 0 \Rightarrow 6x = 6 \Rightarrow \boxed{x = 1}$$

$$\hookrightarrow \boxed{y = \frac{4}{3}}$$

$$\hookrightarrow \boxed{z = 12 - 4 - 4 = 4}$$

$$\boxed{\left(1, \frac{4}{3}, 4\right) = (x, y, z)}$$

2ND DERIV. TEST

$$f_{xx} = -8y$$

$$f_{xx}\left(1, \frac{4}{3}\right) = -\frac{32}{3} < 0$$

$$f_{yy} = -6x$$

$$f_{yy}\left(1, \frac{4}{3}\right) = -6 < 0$$

$$f_{xy} = 12 - 8x - 6y$$

$$f_{xy}\left(1, \frac{4}{3}\right) = 12 - 8 - 8 = -4$$

$$D = \left(-\frac{32}{3}\right)(-6) - (-4)^2 = 64 - 16 = 48 > 0$$

LOCAL
MAX!!!