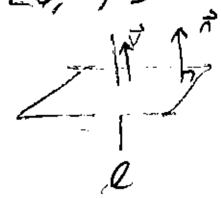
1. (13 pts)

(a) Multiple choice (circle **all** that apply, no work needed):

i. Let ℓ be the line $\mathbf{r}(t) = \langle -3t, 2t+7, 6-t \rangle$ and $\mathcal{P}$ the plane $6x - 4y + 2z = 4$. Then ℓ is... $\vec{v} = \langle -3, 2, -1 \rangle$ IS PARALLEL TO THE NORMAL $\langle 6, -4, 2 \rangle$



(i) orthogonal to $\mathcal{P}$ (ii) parallel to $\mathcal{P}$

(iii) contained in $\mathcal{P}$ (iv) intersects $\mathcal{P}$ at one point

ii. Which of the following vector functions give points that are always on the curve of intersection of $x^2 + z^2 = 4$ and $x = 2y$:

$x \overset{\checkmark}{=} 2y$, $x^2+z^2 = 4t^2 \neq 4$ (i) $\langle 2t, t, 0 \rangle$

$x \overset{\checkmark}{=} 2y$ $x^2+z^2 \overset{\checkmark}{=} 4$ (ii) $\langle 2\cos(t), \cos(t), 2\sin(t) \rangle$

(iii) $\langle 2\sin(t^3), \sin(t^3), 2\cos(t^3) \rangle$ (iv) $\langle t, \frac{1}{2}t, \sqrt{4-t^2} \rangle$

$x \overset{\checkmark}{=} 2y$, $x^2+z^2 \overset{\checkmark}{=} 4$ $x \overset{\checkmark}{=} 2y$, $x^2+z^2 \overset{\checkmark}{=} 4$

(b) A line, L, passes thru $(1, 2, 2)$ and the *center* of the sphere, S, given by $x^2+y^2+z^2-6z = 27$. Find all (x, y, z) points of intersection of the line, L, and the sphere, S.

$$x^2 + y^2 + z^2 - 6z + 9 = 27 + 9$$

$$x^2 + y^2 + (z-3)^2 = 36 \qquad \text{CENTER} = (0,0,3)$$

LINE: $x = 0 + (1-0)t, \; y = 0 + (2-0)t, \; z = 3 + (2-3)t$

$$x = t, \; y = 2t, \; z = 3 - t$$

INTERSECTION: $t^2 + (2t)^2 + (3 - t - 3)^2 \overset{?}{=} 36$

$$t^2 + 4t^2 + t^2 = 36$$

$$6t^2 = 36$$

$$t^2 = 6 \Rightarrow t = \pm\sqrt{6}$$

$$\boxed{(x, y, z) = (-\sqrt{6}, -2\sqrt{6}, 3+\sqrt{6}) \text{ or } (\sqrt{6}, 2\sqrt{6}, 3-\sqrt{6})}$$

2. (12 pts)

(a) Find an equation for the surface consisting of all points that are equidistant from the x-axis and the point $(0,0,1)$ AND give a precise name for the corresponding 3D surface.

"DIST. FROM (x,y,z) TO $\underbrace{\text{X-AXIS}}_{(x,0,0)}$" $\stackrel{?}{=}$ "DIST FROM (x,y,z) TO $(0,0,1)$"

$$\boxed{\sqrt{(x-x)^2+(y-0)^2+(z-0)^2} \stackrel{?}{=} \sqrt{(x-0)^2+(y-0)^2+(z-1)^2}}$$

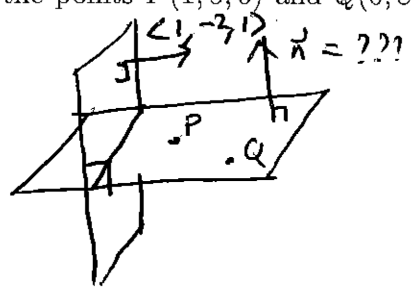
$$\Rightarrow \quad \cancel{y^2} + z^2 \stackrel{?}{=} x^2 + \cancel{y^2} + (z-1)^2$$

$$\cancel{z^2} = x^2 + \cancel{z^2} - 2z + 1$$

$$\boxed{2z = x^2 + 1}$$

SURFACE NAME: $\boxed{\text{PARABOLIC CYLINDER}}$

(b) Find the equation of the plane that contains the points $P(1,5,0)$ and $Q(0,8,2)$ and is orthogonal to the plane $x - 2y + z = 10$.



SINCE $x - 2y + z = 10$
IS ORTHOGONAL TO THE DESIRED
PLANE, $\langle 1,-2,1 \rangle$ MUST
BE PARALLEL TO THE DESIRED
PLANE.

IN ADDITION, $\overrightarrow{PQ} = \langle -1, 3, 2 \rangle$ IS ALSO PARALLEL TO THE
DESIRE PLANE.

THUS,
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -1 & 3 & 2 \end{vmatrix} = (-4-3)\vec{i} - (2-{-1})\vec{j} + (3-2)\vec{k}$$
$$= \langle -7, -3, 1 \rangle$$ IS ORTHOGONAL TO THE DESIRED PLANE.

$$\boxed{-7(x-1) - 3(y-5) + z = 0} \Rightarrow -7x + 7 - 3y + 15 + z = 0$$
$$\Rightarrow -7x - 3y + z = -22$$

OR

$$-7(x-0) - 3(y-8) + (z-2) = 0$$ SAME

OR ANY MULTIPLE (NON-ZERO)

3. (13 pts)

(a) For some curve, $\mathbf{r}(t)$, you are given $\mathbf{T}(0) = \left\langle \frac{3}{5}, 0, -\frac{4}{5} \right\rangle$ = 'unit tangent at $t = 0$'.
(The three parts below are short answer)

i. If $|\mathbf{r}'(0)| = 6$, then give the vector $\mathbf{r}'(0) = 6\vec{T}(0) = \boxed{\left\langle \frac{18}{5}, 0, -\frac{24}{5} \right\rangle}$

ii. If you are told that $\mathbf{N}(0) = \langle a, b, 0 \rangle$ = 'the principal unit normal at $t = 0$' for some numbers a and b, then what can you conclude about a and b? (list all possibilities)

$\vec{T}(0) \cdot \vec{N}(0) = 0$

$\Rightarrow a \cdot \frac{3}{5} + 0 + 0 = 0 \Rightarrow \boxed{a = 0}$

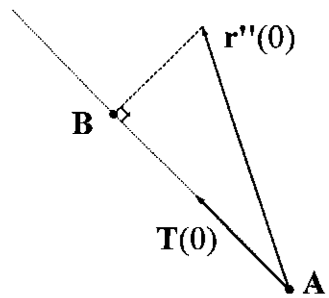
$a = 0$

$b = -1$ or 1

$\langle 0, b, 0 \rangle$ MUST BE A UNIT VECTOR!

iii. If $\mathbf{r}''(0) = \left\langle \frac{5}{4}, 2, -\frac{5}{3} \right\rangle$ is drawn tail-to-tail with $\mathbf{T}(0)$ as shown below. What is the distance from point A to point B in the picture?

$$\frac{\vec{r}'' \cdot \vec{T}}{|\vec{T}|} = \frac{3}{4} + 0 + \frac{4}{3} = \frac{9}{12} + \frac{16}{12} = \boxed{25/12}$$



(b) Find $\mathbf{p}(t)$ if $\mathbf{p}'(t) = \langle 3t^2, 2te^{t^2}, 4te^t \rangle$ and $\mathbf{p}(0) = \langle 0, 0, 0 \rangle$.

$\int 3t^2\,dt = t^3 + c_1 \Rightarrow 0^3 + c_1 = 0 \Rightarrow c_1 = 0$

$\int 2te^{t^2}\,dt$ $\quad u = t^2$, $du = 2t\,dt$

$= \int e^u\,du = e^u + c_2 = e^{t^2} + c_2 \Rightarrow e^{0^2} + c_2 = 0 \Rightarrow c_2 = -1$

$\int 4te^t\,dt$ $\quad u = 4t$, $du = 4\,dt$, $dv = e^t\,dt$, $v = e^t$

$= 4te^t - \int 4e^t\,dt$

$= 4te^t - 4e^t + c_3 \Rightarrow 0 - 4 + c_3 = 0 \quad c_3 = 4$

$$\boxed{\langle t^3, e^{t^2} - 1, 4te^t - 4e^t + 4 \rangle}$$

4. (12 pts)

(a) Find the angle of intersection (to the nearest degree) of the two curves

$$\mathbf{r}_1(t) = \langle t, 6-2t, 15+t^2\rangle \text{ and } \mathbf{r}_2(s) = \langle 5-s, 2s-4, s^2\rangle.$$

$$\left.\begin{array}{l} t \overset{?}{=} 5-s \\ 6-2t \overset{?}{=} 2s-4 \end{array}\right\} \Rightarrow 6-2(5-s) = 2s-4 \Rightarrow 6-10+2s = 2s-4 \Rightarrow 0=0$$

MOVE ON TO THIRD EQUATION

$$15+t^2 \overset{?}{=} s^2 \qquad 15+(5-s)^2 = s^2 \Rightarrow 15+25-10s+\cancel{s^2} = \cancel{s^2}$$

$$\Rightarrow 40 = 10s \Rightarrow \boxed{s=4} \Rightarrow \boxed{t=1}$$

$$\vec{r}_1'(t) = \langle 1, -2, 2t\rangle \qquad \vec{r}_2'(s) = \langle -1, 2, 2s\rangle$$

$$\vec{r}_1'(1) = \langle 1, +2, 2\rangle \qquad \vec{r}_2'(4) = \langle +1, -2, 8\rangle$$

$$\cos(\theta) = \frac{-1+4+16}{\sqrt{1+4+4}\sqrt{1+4+64}} = \frac{11}{3\sqrt{69}} \Rightarrow \theta = \boxed{\cos^{-1}\left(\frac{11}{3\sqrt{69}}\right) \approx 64^\circ}$$

(b) For $x > 0$, find the x-value at which curvature is maximum for $f(x) = \frac{x^3}{3}$.

$$f'(x) = x^2 \quad f''(x) = 2x$$

$$K(x) = \frac{|2x|}{(1+(x^2)^2)^{3/2}} = \frac{2x}{(1+x^4)^{3/2}}$$

$$k'(x) = \frac{(1+x^4)^{3/2}\cdot 2 - \cancel{2}x\,\frac{3}{\cancel{2}}\cdot 4x^3(1+x^4)^{1/2}}{(1+x^4)^3} \overset{?}{=} 0$$

$$\Rightarrow 2(1+x^4)^{1/2}\left[1+x^4-6x^4\right] \overset{?}{=} 0$$

$$\Rightarrow 1-5x^4 \overset{?}{=} 0$$

$$x^4 = 1/5$$

$$\boxed{x = (1/5)^{1/4}}$$