1. (10 points) Compute the following integrals.

(a) $\int_1^e x \ln(x)\, dx$.

$u = \ln(x) \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$\frac{1}{2}x^2 \ln(x)\Big|_1^e - \int_1^e \frac{1}{2}x^2 \frac{1}{x}\, dx$

$\left(\frac{1}{2}e^2 \ln(e) - \frac{1}{2}1^2\ln(1)\right) - \frac{1}{2}\int_1^e x\, dx$

$\frac{1}{2}e^2 - \frac{1}{2}\left[\frac{1}{2}x^2\Big|_1^e\right]$

$\frac{1}{2}e^2 - \frac{1}{4}(e^2 - 1) = \boxed{\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}(e^2+1) \approx 2.0973}$

(b) $\int \sin^2(x)\cos^3(x)\, dx$

$\int \sin^2(x)\cos^2(x)\cos(x)dx \qquad u = \sin(x)$

$\int \sin^2(x)(1-\sin^2(x))\cos(x)dx \qquad du = \cos(x)dx$

$\int u^2(1-u^2)\,du$

$= \int u^2 - u^4\, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

$= \boxed{\frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C}$

2. (10 points) Compute the following integrals.

(a) $\int \frac{e^x}{(e^x)^2 - 4}\,dx$

$u = e^x$

$du = e^x dx$

$\int \frac{1}{u^2 - 4}\,du$

$\int \frac{1}{(u+2)(u-2)}\,du$

$\frac{1}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$

$1 = A(u-2) + B(u+2)$

$\left.\begin{array}{l} u = 2 \Rightarrow B = 1/4 \\ u = -2 \Rightarrow A = -1/4 \end{array}\right\}$ or equating coefficients

$= \int \frac{-1/4}{u+2} + \frac{1/4}{u-2}\,du$

$= -\frac{1}{4}\ln|u+2| + \frac{1}{4}\ln|u-2| + C$

$= \boxed{-\frac{1}{4}\ln|e^x+2| + \frac{1}{4}\ln|e^x-2| + C}$

(b) $\int \frac{1}{(4 - x^2)^{\frac{3}{2}}}\,dx.$

$x = 2\sin(\theta)$

$dx = 2\cos(\theta)d\theta$

$4 - x^2 = 4 - 4\sin^2(\theta)$

$= 4(1 - \sin^2(\theta))$

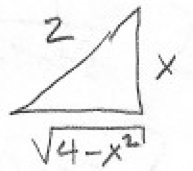
$= 4\cos^2(\theta)$

$\int \frac{1}{(4\cos^2(\theta))^{3/2}} 2\cos(\theta)d\theta$

$\int \frac{1}{(2\cos(\theta))^3} 2\cos(\theta)d\theta$

$\int \frac{1}{(2\cos(\theta))^2}\,d\theta$

$\frac{1}{4}\int \frac{1}{\cos^2(\theta)}\,d\theta$



$\frac{1}{4}\int \sec^2(\theta)d\theta = \frac{1}{4}\tan(\theta) + C$

$= \boxed{\frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C}$

3. (10 points) Compute the following integrals.

(a) $\int \frac{x+2}{x(x+1)^2}\,dx.$

$$\frac{x+2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x+2 = A(x+1)^2 + Bx(x+1) + Cx$$

$$x=0 \Rightarrow \boxed{A=2}$$

$$x=-1 \Rightarrow 1 = -C \quad \boxed{C=-1}$$

$$x=1 \Rightarrow 3 = 2(2)^2 + B(2) - 1$$

$$3 = 8 + 2B - 1$$

$$-4 = 2B \quad \boxed{B=-2}$$

or equate coefficients

$(x+1)^{-2}$

$$\int \frac{2}{x} + \frac{-2}{x+1} + \frac{-1}{(x+1)^2}\,dx$$

$$\boxed{2\ln|x| - 2\ln|x+1| + \frac{1}{x+1} + C}$$

(b) $\int \frac{1}{\sqrt{x^2+8x+7}}\,dx.$

$$\int \frac{1}{\sqrt{x^2+8x+16-16+7}}\,dx = \int \frac{1}{\sqrt{(x+4)^2-9}}\,dx$$

OR DO A U-SUB FIRST

$$x+4 = 3\sec(\theta)$$

$$dx = 3\sec(\theta)\tan(\theta)\,d\theta$$

$$\int \frac{1}{\sqrt{9\sec^2(\theta)-9}}\,3\sec(\theta)\tan(\theta)\,d\theta$$

$$\sqrt{9(\sec^2(\theta)-1)}$$

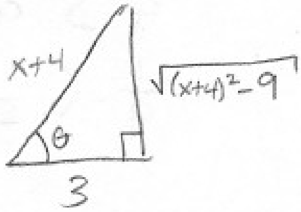
$$= \sqrt{9\tan^2(\theta)} = 3\tan(\theta)$$

$$\int \frac{1}{\cancel{3\tan(\theta)}}\,\cancel{3}\sec(\theta)\,\cancel{\tan(\theta)}\,d\theta$$

$$\sec(\theta) = \frac{x+4}{3} = \frac{\text{hyp}}{\text{adj}}$$

$$\int \sec(\theta)\,d\theta = \ln|\sec(\theta)+\tan(\theta)| + C$$

$$= \boxed{\ln\left|\frac{x+4}{3} + \frac{\sqrt{(x+4)^2-9}}{3}\right| + C}$$



4. (6 points) Use Simpson's rule with $n = 4$ subdivisions to approximate the definite integral

$$\int_2^4 \frac{x}{\ln(x)}\,dx.$$

$\Delta x = \frac{4-2}{4} = \frac{1}{2}$ $\quad x_0 = 2,\ x_1 = 2.5,\ x_2 = 3,$

$x_3 = 3.5,\ x_4 = 4$

$$\int_2^4 \frac{x}{\ln(x)}\,dx \approx \boxed{\frac{1}{3}\cdot\frac{1}{2}\left[\frac{2}{\ln(2)} + 4\frac{2.5}{\ln(2.5)} + 2\frac{3}{\ln(3)} + 4\frac{3.5}{\ln(3.5)} + \frac{4}{\ln(4)}\right]}$$

$$\approx 5.553513434$$

5. (7 points) Determine whether the following integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^\infty xe^{-2x}\,dx.$$

$\lim_{t\to\infty} \int_1^t xe^{-2x}\,dx$ $\qquad u = x \quad dv = e^{-2x}\,dx$

$du = dx \quad v = \frac{1}{-2}e^{-2x}$

$\lim_{t\to\infty} \left[-\frac{1}{2}xe^{-2x}\Big|_1^t + \int_1^t \frac{1}{2}e^{-2x}\,dx\right]$

$\lim_{t\to\infty} \left[-\frac{1}{2}te^{-2t} + \frac{1}{2}e^{-2} - \frac{1}{4}\left(e^{-2x}\Big|_1^t\right)\right]$

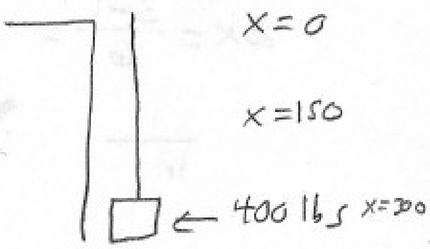
$\lim_{t\to\infty} \left[-\frac{t}{2e^{2t}} + \frac{1}{2e^2} - \left(\cancel{\frac{1}{4e^{2t}}}^{\,0} - \frac{1}{4e^2}\right)\right]$

L'Hopital's

$\lim_{t\to\infty} \left[-\cancel{\frac{1}{4e^{2t}}}^{\,0} + \frac{1}{2e^2} + \frac{1}{4e^2}\right]$

$$= \boxed{\frac{3}{4e^2}}$$

6. (7 points) A cable that weighs 4 pounds per foot is used to lift 400 pounds of coal up a mineshaft. If the mineshaft is 300 feet deep, how much work is required to lift the coal halfway to the top?



$$\int_0^{150} 4x\,dx$$ ← work to lift top half of cable

$$+ \int_{150}^{300} 4\cdot 150\,dx$$ ← work to lift bottom half of cable

$$+ \int_{150}^{300} 400\,dx$$ ← work to lift coal

$$2x^2\Big|_0^{150} = 45000$$

weight of lower half of cable

$$+ \; 600\cdot 150 = 90000$$

$$+ \; 400\cdot 150 = 60000$$

$$\boxed{\text{Work} = 195{,}000 \text{ ft-lbs}}$$