5.3 Fundamental Theorem of Calculus (FTOC)

Recall: FTOC (Part 1)

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

In other words, for any constant a, the "accumulated signed area"

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$.

Entry Tasks:

(a) What does the FTOC say about…

$$F(x) = \int_0^x t^2 dt \quad \text{and} \quad G(x) = \int_{-1}^x t^2 dt$$

(b) A car is traveling at $v(t) = 20$ miles/hr. What does the FTOC say about…

$$d(x) = \int_2^x v(t)dt$$

Mechanically using FTOC (Part 1)

Compute the **derivatives** of the following functions:

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)dt\right) = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x)$$

1. $g(x) = \int_{3}^{x} \cos(t)\, dt$

2. $h(x) = \int_{x}^{-2} te^{t} dt$

3. $f(x) = \int_{0}^{x^3} t + \sin(t)\, dt$

4. $k(x) = \int_{1+x^2}^{x^3} \sqrt{2+t}\, dt$

FTOC (Part 2):

If F(x) <u>any</u> antiderivative of f(x),

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof sketch:

Let $F(x)$ by any antiderivative.

By the FTOC (Part 1), we can think of $F(x)$ as an accumulated area function such as

$$F(x) = \int_c^x f(t)dt$$

for some number *c*.

Then using integral (area) rules we have

$$F(b) - F(a) = \int_c^b f(t)dt - \int_c^a f(t)dt$$

$$= \int_c^b f(t)dt + \int_a^c f(t)dt = \int_a^b f(t)dt$$

Mechanically using FTOC (Part 2)

Evaluate

1. $\int_0^1 x^3 \, dx$

2. $\int_0^\pi \sin(t) \, dt$

3. $\int_{1}^{4} \frac{1}{w}\, dw$

5. $\int_{1}^{2} \frac{3}{x^2}\, dx$

4. $\int_{0}^{5} e^x\, dx$

6. $\int_{1}^{4} \sqrt{x}\, dx$

7. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

8. $\displaystyle\int_0^{\pi/3} \sec(x)\tan(x)\,dx$

5.4 The Indefinite Integral and Net/Total Change

Def'n: The **indefinite integral** of $f(x)$ is defined to be the general antiderivative of $f(x)$.
And we write

$$\int f(x)dx = F(x) + C,$$

where $F(x)$ is any antiderivative of $f(x)$.

Example:

$$\int 6e^x + 4x - 5\sqrt{x}\,dx$$

Net Change and Total Change

The FTOC(2) says the **net change** in $f(x)$ from $x = a$ to $x = b$ is the integral of its **rate**. That is:

$$\int_a^b f'(t)dt = f(b) - f(a)$$

For example

Assume an object is moving along a straight line (up/down or left/right).

$s(t)$ = 'location at time *t*'
$v(t)$ = 'velocity at time t'

pos. $v(t)$ means moving up/right
neg. $v(t)$ means moving down/left

The FTOC (part 2) says

$$\int_a^b v(t)dt = s(b) - s(a)$$

i.e.

'integral of velocity'= '**net change** in dist'

We also call this the *displacement*.

We define **total change** in dist. by

$$\int_a^b |v(t)|dt$$

which we compute by

1. Solving $v(t) = 0$ for t.

2. Splitting up the integral at these t values. Then dropping the absolute value and integrating separately.

3. Adding together as positive numbers.

Example: $v(t) = t^2 - 2t - 8$ ft/sec
Compute the total distance traveled from t = 1 to t = 6.