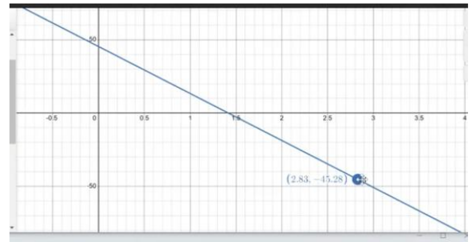
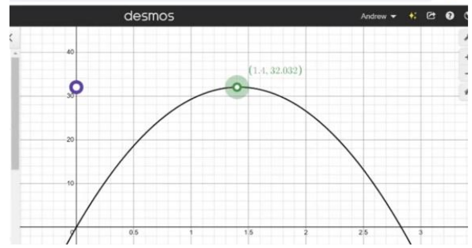
Closing Tues: HW 4.9, 5.1
Closing Thur: HW 5.2, 5.3

4.9 Antiderivatives (continued)

Entry Task:
Assume gravity provides a downward acceleration of 32 ft/s^2.
A ball thrown straight at $t = 0$ sec and returns to your hand at $t = 3$ *sec*.
Find the height of the ball above your hand.





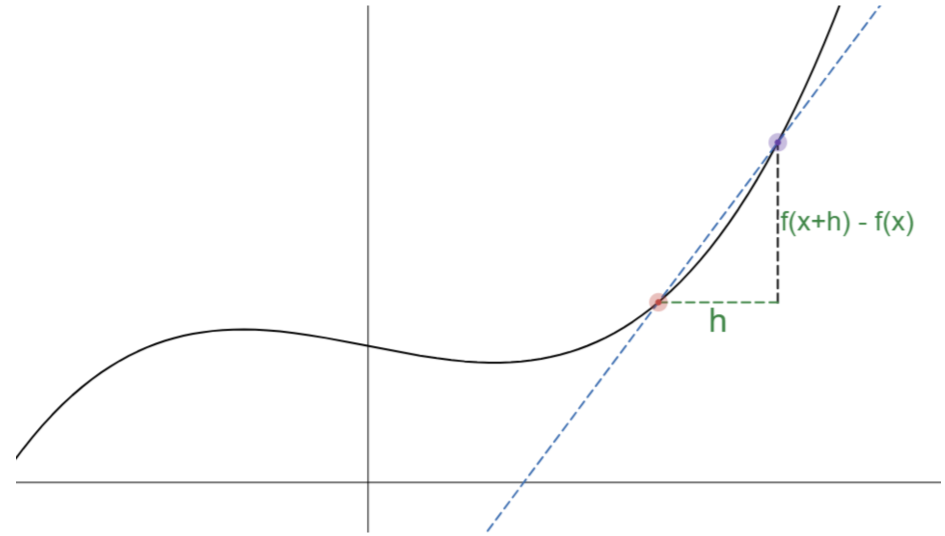
In other words, solve the initial value problem:

$$h''(t) = -32, h(0) = 0, h(3) = 0$$

Clip: https://youtube.com/shorts/iqj3byvqLmc?feature=share
Details: https://www.youtube.com/watch?v=07TDV5SgJT4

5.1 Defining Area (Riemann sums)

Calculus is based on limiting processes that "approach" the exact answer to a rate question.

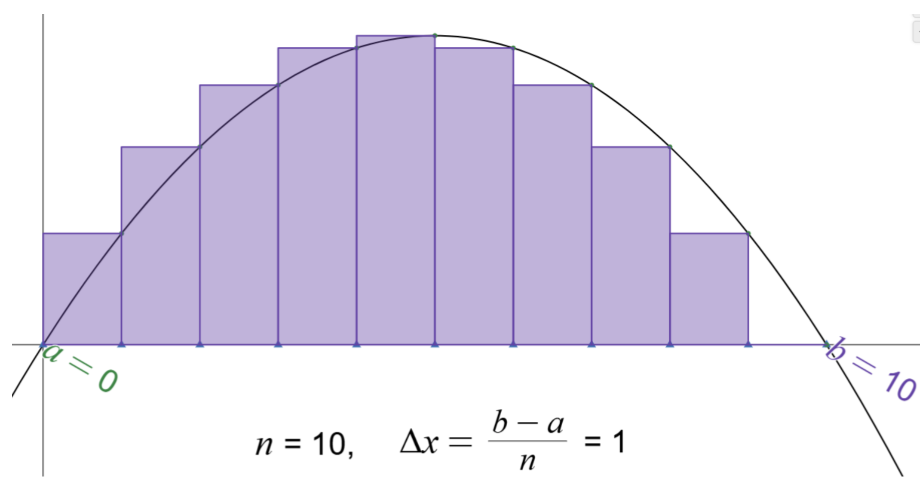


Desmos: https://www.desmos.com/calculator/e11b1c2af2

In Calculus I, you defined

f'(x) = `slope of the tangent at *x*'

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



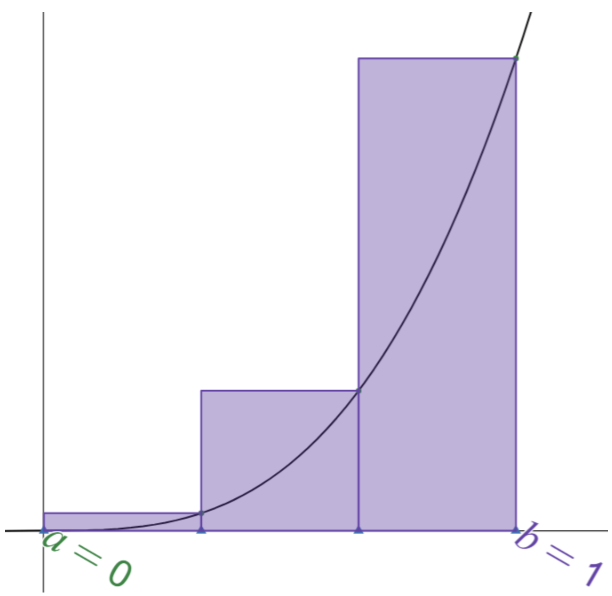
Desmos: https://www.desmos.com/calculator/n6ipjngyos

In Calculus II, we will see that antiderivatives are related to the area `under' a graph

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f({x_i}^*)\Delta x$$

Riemann sums set up:
A procedure to get better and better approximations of the area "under" $f(x)$.

1. Break into *n* subintervals.
   $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$
2. Draw *n* rectangles.
   Area of each rectangle =
   (height)(width) $= f(x_i^*)\Delta x$
3. Add up rectangle areas.



Desmos: https://www.desmos.com/calculator/n6ipjngyos

Example:
Approx. the area under $f(x) = x^3$ from x = 0 to x = 1 using n = 3 subdivisions and *right-endpoints* to find the heights.

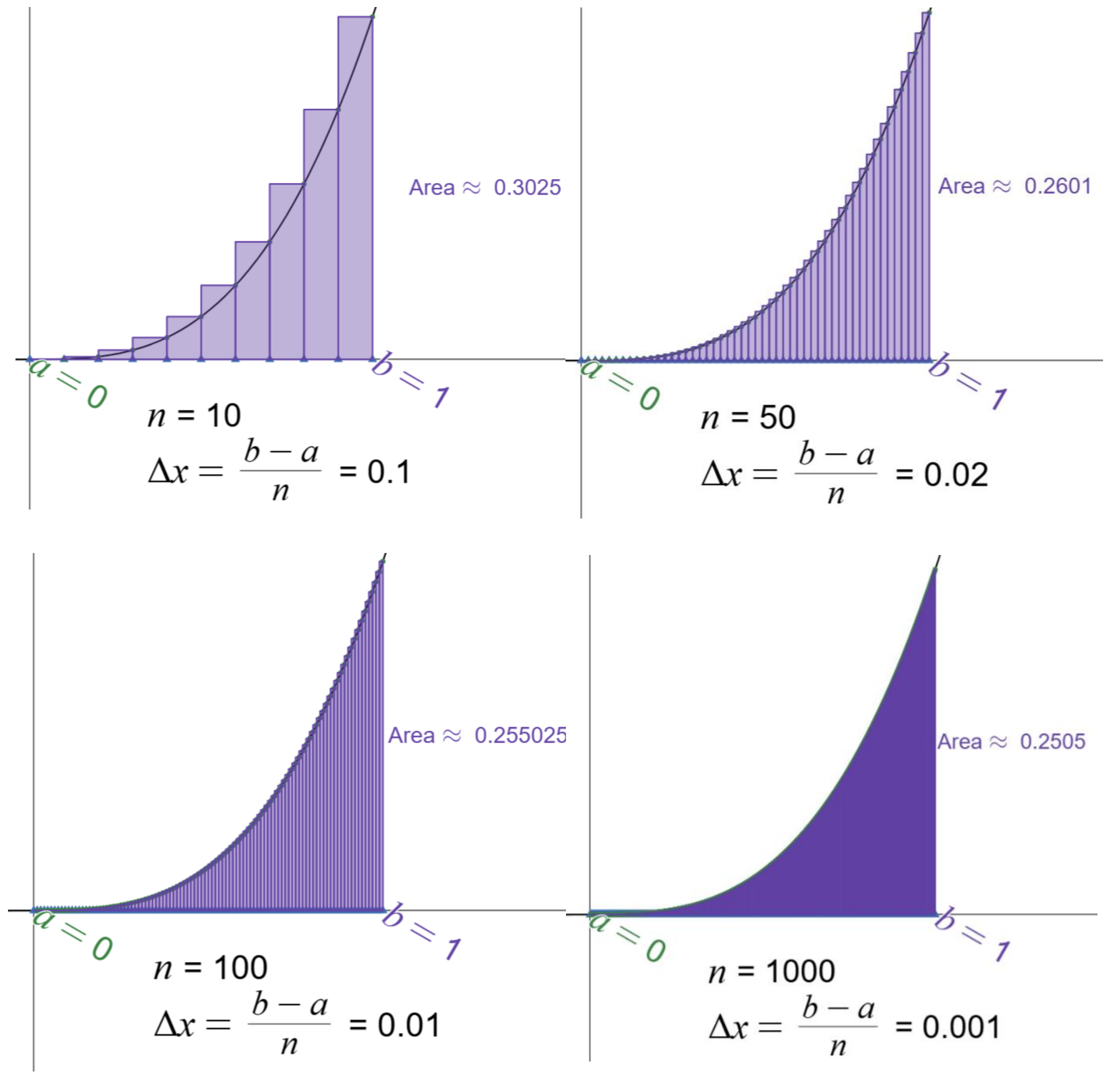
You try:

Approx. the area under $f(x) = x^3$ from x = 0 to x = 1 using n = 4 subdivisions and *right-endpoints* to find the heights.

I did this again with 100 subdivisions, then 1000, then 10000.

Here is a summary of my findings:

n	R_n	L_n
4	0.390625	0.140625
5	0.36	0.16
10	0.3025	0.2025
100	0.255025	0.245025
1000	0.25050025	0.24950025
10000	0.2499500025	0.2500500025



General Pattern: (right-endpoint)

For $f(x) = x^3$ on x = 0 to x = 1 with "n" subdivisions…

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}, \; x_i = 0 + i\frac{1}{n} = \frac{i}{n}$$

$$\text{Height} = f(x_i) = x_i^3 = \left(\frac{i}{n}\right)^3$$

$$\text{Area} = f(x_i)\Delta x = x_i^3 \Delta x = \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

Adding up the rectangle areas

$$\text{Sum} = \sum_{i=1}^{n} x_i^3 \Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

$$\text{Exact Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

Another Example:

Using sigma notation, write down the general Riemann sum definition of the area from x = 5 to x = 7 under

$$f(x) = 3x + \sqrt{x}$$

$$\Delta x = \frac{b-a}{n} =$$

$$x_i = a + i\,\Delta x =$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x =$$

Application Note

Velocity/Distance & Reimann Sums

When velocity is a ***constant***:

Distance = Velocity · Time

Example:

You are accelerating in a car. You get the following measurements:

t (sec)	0	0.5	1.0	1.5	2.0
v(t) (ft/s)	0	6.2	10.8	14.9	18.1

Estimate the distance traveled by the car traveled from 0 to 2 seconds. *Same idea!*

	Low Estimate	High Estimate
0 to 0.5	$0\frac{ft}{s}$·0.5 s = **0 ft**	$6.2\frac{ft}{s}$·0.5 s = **3.1 ft**
0.5 to 1.0	$6.2\frac{ft}{s}$·0.5 s = **3.1 ft**	$10.8\frac{ft}{s}$·0.5 s = **5.4 ft**
1.0 to 1.5	$10.8\frac{ft}{s}$·0.5 s = **5.4 ft**	$14.9\frac{ft}{s}$·0.5 s = **7.45 ft**
1.5 to 2.0	$14.9\frac{ft}{s}$·0.5 s = **7.45 ft**	$18.1\frac{ft}{s}$·0.5 s = **9.05 ft**
Total	**15.95 ft**	**25 ft**

All we are doing is multiplying and adding. The ***units*** of

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

will be **units of f(x) times units of x**.

5.2 The Definite Integral

Def'n:

We define the **definite integral of f(x) from x = a to x = b** by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x\,,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Basic Integral Rules:

1. $\int_{a}^{b} c\, dx = (b-a)c$

2. $\int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx = \int_{a}^{c} f(x)dx$

Examples:

1. $\int_{4}^{10} 5\, dx =$

2. $\int_{0}^{3} x^2 dx + \int_{3}^{7} x^2 dx =$

Basic Integral Rules (continued):

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)dx$

and

$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)dx + \int_a^b g(x)\,dx$

4. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

Examples:

3. $\int_0^4 5x + 3\,dx =$

4. $\int_3^1 x^3 dx = -\int_1^3 x^3\,dx$

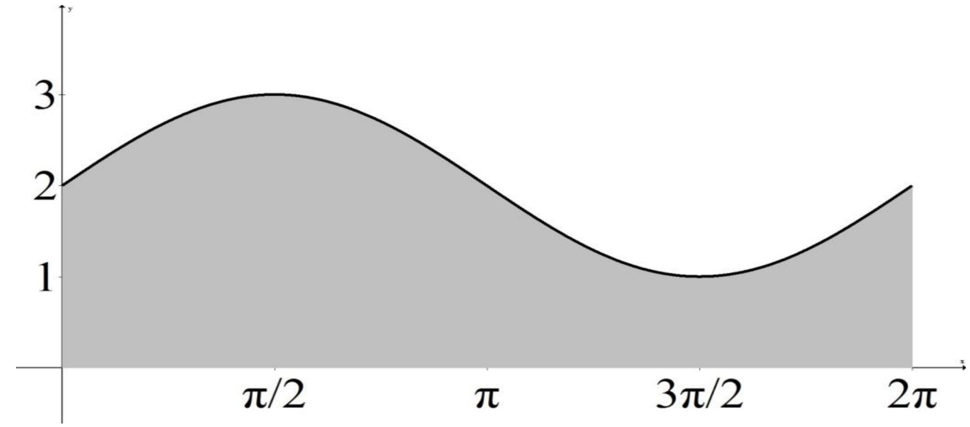
Note on quick estimates (HW 5.2: 10,11)

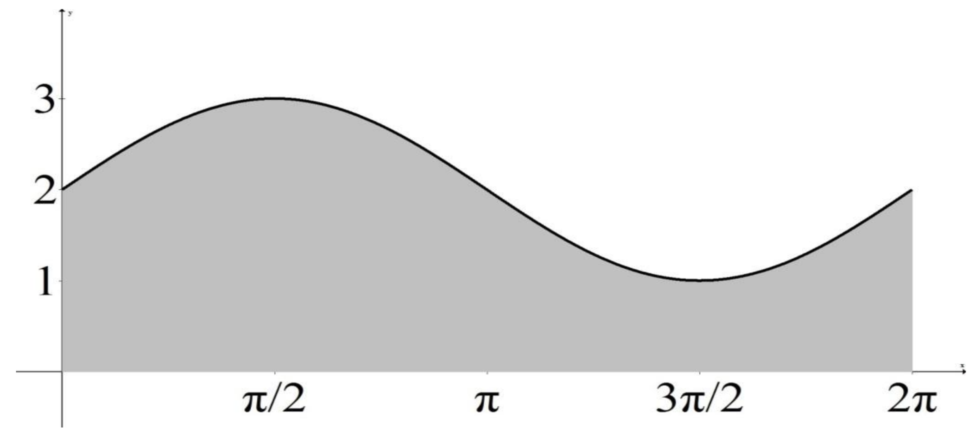
$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Example: Consider the area under

f(x) = sin(x) + 2

on the interval *x = 0* to *x = 2π.*





(a) What is the max of f(x)? (label M)
(b) What is the min of f(x)? (label m)

(c) Draw **one** rectangle that completely contains all the shaded area?
(d) Draw **one** rectangle that is completely inside the shaded area?
What can you conclude?