1. (12 pts) Evaluate

(a) $\int \frac{9x^2}{(x-1)^2(x+2)}dx$

$$\frac{9x^2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\Rightarrow 9x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$x=1 \Rightarrow 9 = B(3) \Rightarrow B = 3$$

$$x=-2 \Rightarrow 36 = C(9) \Rightarrow C = 4$$

COEF. OF x^2: $A + C \overset{?}{=} 9$

$$\Rightarrow A = 9 - C = 5$$

$$= \int \frac{5}{x-1} + \frac{3}{(x-1)^2} + \frac{4}{x+2}\,dx$$

$$= \boxed{5\ln|x-1| - \frac{3}{x-1} + 4\ln|x+2| + C}$$

(b) $\int \tan^{10}(2x)\sec^4(2x)\ dx$

$t = 2x$
$dt = 2dx$

$$= \frac{1}{2}\int \tan^{10}(t)\sec^4(t)\,dt$$

$$= \frac{1}{2}\int \tan^{10}(t)\sec^2(t)\sec^2(t)\,dt$$

$u = \tan(t)$
$du = \sec^2(t)\,dt$

$$= \frac{1}{2}\int \tan^{10}(t)(\tan^2(t)+1)\sec^2(t)\,dt$$

$$= \frac{1}{2}\int u^{10}(u^2+1)\,du$$

$$= \frac{1}{2}\int u^{12} + u^{10}\,du$$

$$= \frac{1}{2}\left[\frac{1}{13}u^{13} + \frac{1}{11}u^{11}\right] + C$$

$$= \boxed{\frac{1}{26}\tan^{13}(2x) + \frac{1}{22}\tan^{11}(2x) + C}$$

2. (12 pts) Evaluate

(a) $\int_0^{\pi^2/16} \sin(\sqrt{x})\, dx$

$t = \sqrt{x} \qquad t^2 = x$

$2t\,dt = dx$

$$= \int_0^{\pi/4} 2t \sin(t)\,dt$$

$u = 2t \qquad dv = \sin(t)\,dt$

$du = 2\,dt \qquad v = -\cos(t)$

$$= -2t\cos(t)\Big|_0^{\pi/4} - -\int_0^{\pi/4} 2\cos(t)\,dt$$

$$= -2\,\tfrac{\pi}{4}\,\tfrac{\sqrt{2}}{2} + \left(2\sin(t)\Big|_0^{\pi/4}\right)$$

$$\boxed{= -\frac{\pi\sqrt{2}}{4} + 2\frac{\sqrt{2}}{2} = -\frac{\pi\sqrt{2}}{4} + \sqrt{2} = \sqrt{2}\left(-\tfrac{\pi}{4} + 1\right) = \sqrt{2}\left(\frac{4-\pi}{4}\right)} = \frac{4-\pi}{2\sqrt{2}} \approx 0.303493$$

(b) $\int \frac{1}{(x^2+6x+25)^{3/2}} dx$

$x^2 + 6x + 9 - 9 + 25 = (x+3)^2 + 16$

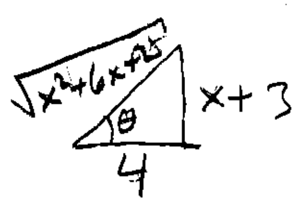
$x + 3 = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

$$\int \frac{1}{(16\sec^2\theta)^{3/2}} 4\sec^2\theta\, d\theta$$

$$\frac{4}{48}\int \frac{1}{\sec\theta}\, d\theta$$

$$\frac{1}{16}\int \cos\theta\, d\theta$$



$$\frac{1}{16}\sin\theta + C$$

$$\boxed{\frac{1}{16}\,\frac{x+3}{\sqrt{x^2+6x+25}} + C}$$

3. (12 points) Evaluate

(a) $\int_0^1 \frac{1}{e^x+1}\,dx$

$$t = e^x$$
$$dt = e^x\,dx$$
$$\frac{1}{e^x}\,dt = dx$$
$$\uparrow$$
$$\frac{1}{t}\,dt = dx$$

$$\int_1^e \frac{1}{(t+1)}\,\frac{1}{t}\,dt$$

$$= \int_1^e \frac{-1}{t+1} + \frac{1}{t}\,dt$$

$$\frac{1}{(t+1)t} = \frac{A}{t+1} + \frac{B}{t}$$

$$= -\ln|t+1| + \ln|t| \Big|_1^e$$

$$= \left(-\ln|e+1| + \ln|e|\right) - \left(-\ln(2) + \ln(1)\right)$$

$$\Rightarrow \quad 1 = At + B(t+1)$$
$$t = 0 \Rightarrow B = 1$$
$$t = -1 \Rightarrow A = -1$$

$$= \boxed{-\ln(e+1) + 1 + \ln(2)}$$

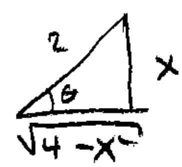
$$= 1 + \ln\left(\frac{2}{e+1}\right) \approx 0.37989$$

(b) $\int \sqrt{4-x^2}\,dx$

$$x = 2\sin\theta$$
$$dx = 2\cos\theta\,d\theta$$

$$\int 2\cos\theta \; 2\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta = 4\int \frac{1}{2}(1+\cos(2\theta))\,d\theta$$



$$= 2\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

$$= 2\theta + \sin(2\theta) + C$$

$$= 2\theta + \underbrace{2\sin\theta\cos\theta}_{} + C$$

$$= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) + 2\,\frac{x}{2}\,\frac{\sqrt{4-x^2}}{2} + C}$$

$$= \boxed{2\sin^{-1}(x/2) + \frac{1}{2}x\sqrt{4-x^2} + C}$$

4. (14 pts) For your answers below, simplify and give exact form.

(a) Find the average value of $4x\ln(x+1)$ from $x=0$ to $x=2$.

$$\frac{1}{2-0}\int_0^2 4x\ln(x+1)\,dx$$

$u=\ln(x+1)$ $\quad dv=2x\,dx$

$du=\frac{1}{x+1}dx$ $\quad v=x^2$

$$=\int_0^2 2x\ln(x+1)\,dx$$

$$=x^2\ln(x+1)\Big|_0^2-\int_0^2\frac{x^2}{x+1}\,dx$$

$$\begin{array}{r} x-1 \\ x+1\;\overline{)\,x^2} \\ -(x^2+x) \\ \hline -x \\ -(-x-1) \\ \hline 1 \end{array}$$

$$=4\ln(3)-\int_0^2 x-1+\frac{1}{x+1}\,dx$$

$$=4\ln(3)-\left[\tfrac{1}{2}x^2-x+\ln|x+1|\,\Big|_0^2\right]$$

$$=4\ln(3)-[2-2+\ln(3)]$$

$$=\boxed{3\ln(3)}=\ln(27)\approx 3.2958$$

(b) Determine if the improper integral below converges or diverges. If it converges, then give the value. You MUST correctly show your work. $\displaystyle\int_0^\infty \frac{1}{x^2+2x+2}\,dx$. (Hint: The denominator is irreducible!)

$$\lim_{t\to\infty}\left[\int_0^t \frac{1}{x^2+2x+2}\,dx\right]$$

$x^2+2x+1-1+2$

$=(x+1)^2+1$

$u=x+1$

$du=dx$

$$\lim_{t\to\infty}\left[\int_1^{t+1}\frac{1}{u^2+1}\,du\right]$$

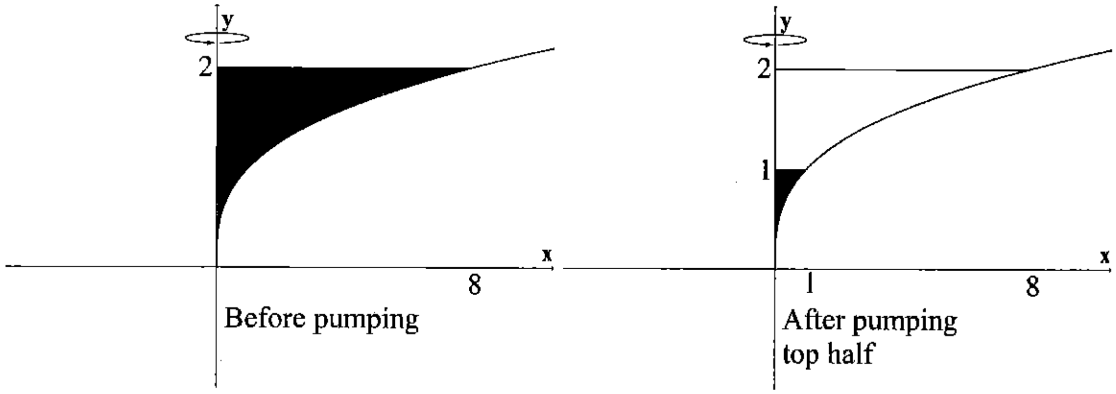
$$=\lim_{t\to\infty}\left[\tan^{-1}(u)\Big|_1^{t+1}\right]$$

$$=\lim_{t\to\infty}\left[\underbrace{\tan^{-1}(t+1)}_{\pi/2}-\underbrace{\tan^{-1}(1)}_{\pi/4}\right]$$

$$=\pi/2-\pi/4$$

$$=\boxed{\pi/4}\approx 0.78540$$

5. (12 points) The region bounded by $y = \sqrt[3]{x}$, $x = 0$, and $y = 2$ is rotated about the y-axis to form a container. Distances are measured in feet. Initially, the tank is full of a fluid that has a density of 20 lbs/ft^3. How much work is done to pump the top 1 foot of liquid out of the container?



$$\int_1^2 (2-y)\, 20\pi (y^3)^2\, dy$$

$$20\pi \int_1^2 (2-y) y^6\, dy$$

$$20\pi \int_1^2 2y^6 - y^7\, dy$$

$$20\pi \left[\frac{2}{7} y^7 - \frac{1}{8} y^8 \Big|_1^2 \right]$$

$$20\pi \left[\left(\frac{2}{7}(2)^7 - \frac{1}{8}(2)^8 \right) - \left(\frac{2}{7} - \frac{1}{8} \right) \right]$$

$$\approx \boxed{277.13 \text{ ft-lbs}}$$