1. (10 pts) Evaluate the integrals.

(a) $\displaystyle\int \frac{x^3(1-x^{5/2})}{\sqrt{x}} + e^{5x-2}\,dx$

SIMPLIFY

$\int \frac{x^3}{x^{1/2}} - \frac{x^{11/2}}{x^{1/2}} + e^{3x-2}\,dx$

$\int x^{5/2} - x^5 + e^{3x-2}\,dx$

EITHER BY GUESS / CHECK OR BY SUBSTITUTION WITH $u = 3x-2$

$\boxed{\frac{2}{7}x^{7/2} - \frac{1}{6}x^6 + \frac{1}{3}e^{3x-2} + C}$

(b) $\displaystyle\int_{1/2}^{e/2} \frac{\sec(\ln(2x))\tan(\ln(2x))}{x}\,dx$

$u = \ln(2x)$

$du = \frac{2}{2x}\,dx$

$dx = x\,du$

$x = 1/2 \rightarrow u = \ln(1) = 0$

$x = e/2 \rightarrow u = \ln(e) = 1$

$\int_0^1 \sec(u)\tan(u)\,du$

$= \sec(u)\Big|_0^1$

$= \sec(1) - \sec(0)$

$\sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1$

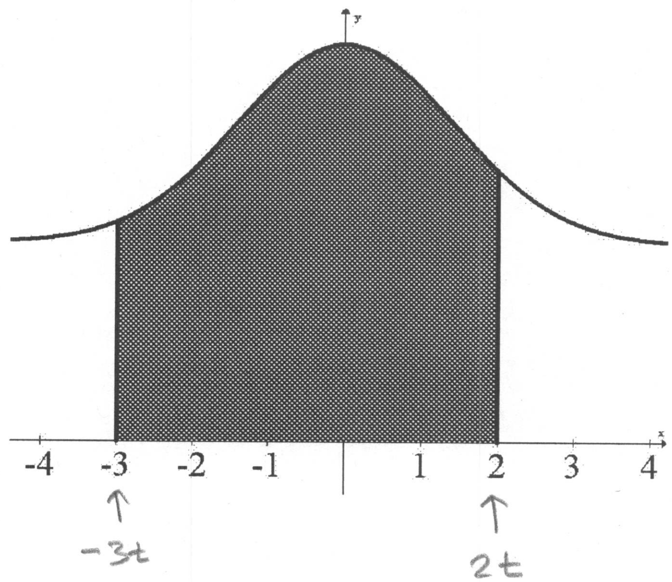
$= \boxed{\sec(1) - 1}$

≈ 0.850816

2. (8 pts) A long wall is in the shape of region between $f(x) = 1 + e^{(-x^2/4)}$ and the y-axis. Two painters start at the origin. One moves in the positive x-direction at a constant rate of 2 feet/minute. The other moves in the negative x-direction at a constant rate of 3 feet/minute. They paint the region of the fall in front of them as they go. (The picture below depicts the painted region after $t = 1$ minute).

(a) (3 pts)

Set up an integral (DO NOT EVALUATE) that gives the area of the wall painted region of the wall after t minutes. (Hint: Your answer will contain the variable t).



$$\boxed{A(t) = \int_{-3t}^{2t} 1 + e^{(-x^2/4)}\,dx}$$

(b) (5 pts) At what **rate** is the area of the painted region changing at $t = 2$ minutes?

$$A(t) = \int_{-3t}^{0} 1 + e^{(-x^2/4)}\,dx + \int_{0}^{2t} 1 + e^{(-x^2/4)}\,dx$$

$$\Rightarrow A(t) = -\int_{0}^{-3t} 1 + e^{(-x^2/4)}\,dx + \int_{0}^{2t} 1 + e^{(-x^2/4)}\,dx$$

$$A'(t) = -(-3)\left(1 + e^{-\frac{9t^2}{4}}\right) + (2)\left(1 + e^{-4t^2/4}\right)$$

$$A'(2) = 3\left(1 + e^{-9}\right) + 2\left(1 + e^{-4}\right)$$

$$= \boxed{5 + 3e^{-9} + 2e^{-4} \quad \frac{ft^2}{min}}$$

$$= 5 + \frac{3}{e^9} + \frac{2}{e^4} \approx 5.0370015$$

3. (~~7~~ pts) Evaluate $\displaystyle\int_{\pi/4}^{3\pi/4} \left|\cos(x)\sin^2(x)\right|\, dx$

$\cos(x)\sin^2(x) \overset{?}{=} 0$

when $x = \ldots, -\pi, 0, \pi, 2\pi, \ldots$

and $x = \ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

only one in region

$\int \cos(x)\sin^2(x)dx$ $\quad u = \sin(x)$, $du = \cos(x)dx$

$= \int u^2 du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\sin^3(x) + C}$

So

$\int_{\pi/4}^{\pi/2} \cos(x)\sin^2(x)dx = \frac{1}{3}\sin^3(x)\Big|_{\pi/4}^{\pi/2} = \frac{1}{3}\left(1 - \left(\frac{\sqrt{2}}{2}\right)^3\right) > 0$

$\int_{\pi/2}^{3\pi/4} \cos(x)\sin^2(x)dx = \frac{1}{3}\sin^3(x)\Big|_{\pi/2}^{3\pi/2} = \frac{1}{3}\left(\left(\frac{\sqrt{2}}{2}\right)^3 - 1\right) < 0$

$\int_{\pi/4}^{3\pi/4} |\cos(x)\sin^2(x)|dx = \boxed{\frac{2}{3}\left(1 - \left(\frac{\sqrt{2}}{2}\right)^3\right)}$ $= \frac{2}{3}\left(1 - \frac{\sqrt{2}}{4}\right) = \frac{2}{3}\left(1 - \frac{1}{2\sqrt{2}}\right) \approx 0.4309644$

4. (8 pts) Consider the integral $\displaystyle\int_1^7 \sin(\sqrt{x})\, dx$.

(a) Approximate this integral using the right endpoint method with $n = 4$ subdivisions. Show your work by writing out all the terms of the sum, then give the decimal value of the approximation.

$\Delta x = \frac{7-1}{4} = \frac{3}{2}$: $x_0 = 1$, $x_1 = \frac{5}{2}$, $x_2 = 4$, $x_3 = \frac{11}{2}$, $x_4 = 7$

$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$

$= \sin\left(\sqrt{\frac{5}{2}}\right)\frac{3}{2} + \sin(\sqrt{4})\frac{3}{2} + \sin\left(\sqrt{\frac{11}{2}}\right)\frac{3}{2} + \sin(\sqrt{7})\frac{3}{2}$

$\approx \boxed{4.649772657}$

(b) In terms of n in general, write out the formal Reimann sum definition (involving a limit and sigma notation) using the right endpoint method for the integral.

$\Delta x = \frac{7-1}{n} = \frac{6}{n}$, $x_i = 1 + \frac{6i}{n}$

$\int_1^7 \sin(\sqrt{x})dx = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\sqrt{1 + \frac{6i}{n}}\right)\frac{6}{n}}$ $= 5.0034465$

5. (15 points) Consider the region, R, bounded by the curve $y = 2x$ and $y = x^2$.

(a) (5 pts) Give the area of the region.

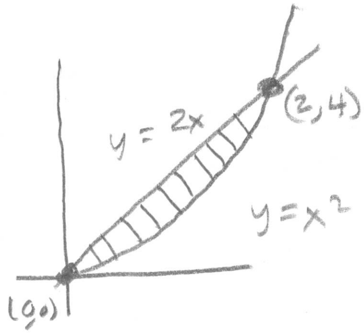
$$\int_0^2 2x - x^2 \, dx \quad \text{or} \quad \int_0^4 \sqrt{y} - \tfrac{1}{2}y \, dy$$

$$= x^2 - \tfrac{1}{3}x^3 \Big|_0^2 \qquad = \tfrac{2}{3}y^{3/2} - \tfrac{1}{4}y^2 \Big|_0^4$$

$$= 4 - \tfrac{8}{3} = \boxed{4/3} \qquad = \tfrac{2}{3}8 - \tfrac{1}{4}16$$

$$= \tfrac{16}{3} - 4 = \boxed{4/3}$$

$1.\overline{3}$ $\qquad$ $1.\overline{3}$



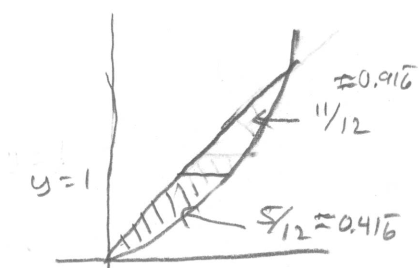
(b) (5 pts) Does the line $y = 1$ divide the region R in half? (Justify with at least one integral calculation).

$$\int_0^1 \sqrt{y} - \tfrac{1}{2}y \, dy = \tfrac{2}{3}y^{3/2} - \tfrac{1}{4}y^2 \Big|_0^1$$

$$= \tfrac{2}{3} - \tfrac{1}{4} = \tfrac{8}{12} - \tfrac{3}{12}$$

$$= \tfrac{5}{12} \neq \tfrac{1}{2}\left(\tfrac{4}{3}\right)$$

$\boxed{\text{NO}}$, $y=1$ does not divide the region in half



(c) (5 pts) Set up AND evaluate and integral for the volume of the solid obtained by rotating R about the x-axis.

$$\int_0^2 \pi(2x)^2 - \pi(x^2)^2 \, dx$$

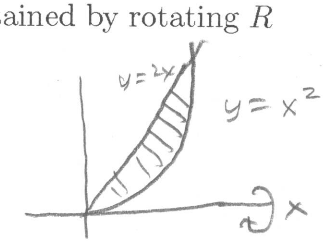
$$\pi \int_0^2 4x^2 - x^4 \, dx$$

$$\pi \left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5 \Big|_0^2\right]$$

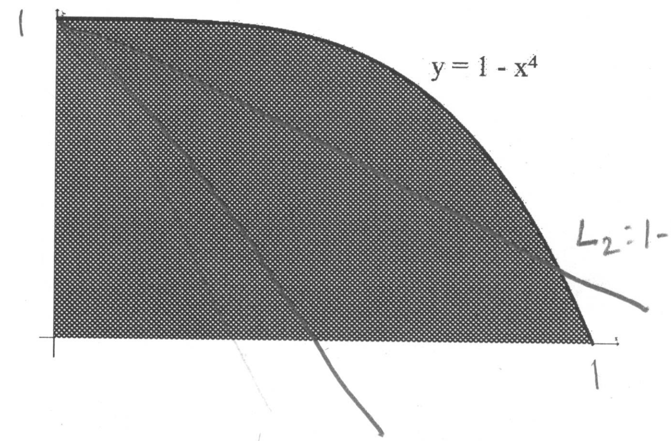
$$\tfrac{5}{15} - \tfrac{3}{15} = \tfrac{2}{15}$$

$$\pi\left[\tfrac{4}{3}\cdot 8 - \tfrac{1}{5}\cdot 32\right] = \pi\left[\tfrac{32}{3} - \tfrac{32}{5}\right] = 32\pi\left[\tfrac{1}{3} - \tfrac{1}{5}\right]$$

$$= \boxed{\frac{64\pi}{15}} \approx 13.40412866$$



6. (12 pts) Dr. Loveless is baking again. He has made a cake in shape of the region bounded in the **first quadrant** by $y = 1 - x^4$ (shown in the picture). He says if you can cut the cake in thirds, he'll eat a third, you'll eat a third and you can throw the last third in his face. BUT, you must cut it using two lines of the form $y = 1 - ax$ and $y = 1 - bx$. Find a and b.



TOTAL AREA

$$= \int_0^1 1 - x^4\,dx$$
$$= x - \frac{1}{5}x^5 \Big|_0^1$$
$$= 1 - \frac{1}{5} = \frac{4}{15}$$

$$\boxed{\frac{1}{3} \text{ of area} = \frac{4}{15}}$$

L1 AREA $= \frac{1}{2}$ BASE $\cdot$ HEIGHT

$\frac{4}{15} = \frac{1}{2}$ BASE $\cdot 1 \Rightarrow$ BASE $= \frac{8}{15}$

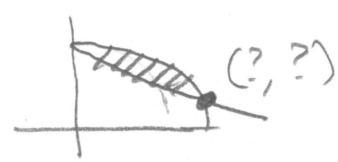
So slope $= -\frac{1}{8/15} = -\frac{15}{8}$ $\Big\}$ $\boxed{a = \frac{15}{8} = 1.875}$

L2 NOTE: MUST BE AS DEPICTED ABOVE BECAUSE THIS AREA $= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ IS LESS THAN $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$

TOTAL

INTERSECT: $1 - bx = 1 - x^4$

$\Rightarrow b = x^3 \Rightarrow x = b^{1/3}$ and $y = 1 - b^{4/3}$



$$\int_0^{b^{1/3}} (1-x^4) - (1-bx)\,dx = \int_0^{b^{1/3}} -x^4 + bx\,dx$$
$$= -\frac{1}{5}x^5 + \frac{b}{2}x^2 \Big|_0^{b^{1/3}}$$
$$= -\frac{1}{5}b^{5/3} + \frac{b}{2}b^{2/3} = b^{5/3}\underbrace{\left(-\frac{1}{5} + \frac{1}{2}\right)}_{3/10} =$$
$$= \frac{3}{10}b^{5/3} \overset{?}{=} \frac{4}{15}$$

$$b^{5/3} = \frac{4 \cdot 10}{15 \cdot 3} = \frac{8}{9} \Rightarrow \boxed{b = \left(\frac{8}{9}\right)^{3/5}} \approx 0.9317694917$$