1. (12 pts) Evaluate

(a) $\int \frac{x-1}{x^3-2x^2}dx$

$$\frac{x-1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$\Rightarrow x-1 = Ax(x-2) + B(x-2) + Cx^2$$

$$x=0 \Rightarrow -1 = B(-2) \Rightarrow B = \tfrac{1}{2}$$

$$x=2 \Rightarrow 1 = C(4) \Rightarrow C = \tfrac{1}{4}$$

MATCHING COEF. OF x^2

$$\left\{\begin{array}{l} \text{LEFT} = 0 \\ \text{RIGHT} = A+C \end{array}\right\} \Rightarrow A+C=0$$

$$A = -C = -\tfrac{1}{4}$$

$$= \int \frac{-1/4}{x} + \frac{1/2}{x^2} + \frac{1/4}{x-2}\,dx$$

$$= \boxed{-\frac{1}{4}\ln|x| - \frac{1}{2x} + \frac{1}{4}\ln|x-2| + C} = \frac{1}{4}\ln\left|\frac{x-2}{x}\right| - \frac{1}{2x} + C$$

(b) $\int x\tan^{-1}(x)\,dx$

$$u = \tan^{-1}(x) \qquad dv = x\,dx$$

$$du = \frac{1}{x^2+1}dx \qquad v = \frac{1}{2}x^2$$

$$= \frac{1}{2}x^2\tan^{-1}(x) - \int \frac{1}{2}\frac{x^2}{(x^2+1)}dx$$

$$\begin{array}{r|l} & 1 \\ x^2+1 & x^2 \\ & -(x^2+1) \\ \hline & -1 \end{array}$$

$$= \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int 1 - \frac{1}{x^2+1}dx$$

$$= \boxed{\frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1}(x) + C}$$

$$= \frac{1}{2}(x^2+1)\tan^{-1}(x) - \frac{1}{2}x + C$$

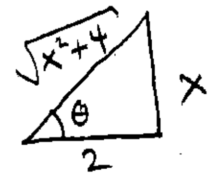
2. (12 pts) Evaluate

(a) $\int \frac{x^2}{(x^2+4)^{3/2}}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$x^2+4 = 4\sec^2\theta$



$$\int \frac{4\tan^2(\theta)}{(4\sec^2\theta)^{3/2}}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{\cancel{8}\tan^2\theta \cancel{\sec^2\theta}}{\cancel{8}\sec^{\cancel{3}}\theta}\, d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\, d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}\, d\theta \qquad \frac{1}{\sec\theta} = \cos\theta$$

$$= \int \sec\theta - \cos\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$= \boxed{\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - \frac{x}{\sqrt{x^2+4}} + C}$$

$$= \ln|\sqrt{x^2+4} + x| - \frac{x}{\sqrt{x^2+4}} \underbrace{-\ln(2) + C}_{D}$$

(b) $\int \frac{\ln(x)}{x^5}dx$ $= \int x^{-5}\ln(x)\,dx$

$u = \ln(x)$ $\quad dv = x^{-5}dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{1}{-4}x^{-4}$

$$= -\frac{1}{4}x^{-4}\ln(x) - \underbrace{\int -\frac{1}{4}x^{-5}dx}_{+\int \frac{1}{4}x^{-5}dx}$$

$$= \boxed{-\frac{1}{4}x^{-4}\ln(x) - \frac{1}{16}x^{-4} + C}$$

$$= -\frac{\ln(x)}{4x^4} - \frac{1}{16x^4} + C$$

$$= -\frac{(4\ln(x) + 1)}{16x^4} + C$$

3. (12 points) Evaluate

(a) $\int x^2 \sin^2(x^3)\cos^2(x^3)\,dx$

$$t = x^3$$
$$dt = 3x^2\,dx$$
$$\frac{1}{3x^2}\,dt = dx$$

$$\int \cancel{x^2} \sin^2(t)\cos^2(t)\frac{1}{3\cancel{x^2}}\,dt$$

$$\frac{1}{3}\int \frac{1}{2}(1-\cos(2t))\frac{1}{2}(1+\cos(2t))\,dt$$

$$\frac{1}{12}\int 1-\cos^2(2t)\,dt = \frac{1}{12}\int 1-\frac{1}{2}(1+\cos(4t))\,dt$$
$$= \frac{1}{12}\int \tfrac{1}{2} - \frac{1}{2}\cos(4t)\,dt$$
$$= \frac{1}{24}\left[t - \frac{1}{4}\sin(4t)\right] + C$$
$$= \boxed{\frac{1}{24}\left(x^3 - \frac{1}{4}\sin(4x^3)\right) + C}$$
$$= \frac{1}{24}x^3 - \frac{1}{96}\sin(4x^3) + C$$

(b) $\int \frac{x}{x^2+6x+13}\,dx$

IRREDUCIBLE

$$x^2+6x+13 = x^2+6x+9-9+13$$
$$= (x+3)^2+4$$

$$= \int \frac{x}{(x+3)^2+4}\,dx$$

$$t = x+3 \qquad t-3 = x$$
$$dt = dx$$

$$= \int \frac{t-3}{t^2+4}\,dt$$

$$= \int \frac{t}{t^2+4}\,dt - \int \frac{3}{t^2+4}\,dt$$

$$= \frac{1}{2}\ln(t^2+4) - \frac{3}{2}\tan^{-1}(t/2) + C$$

$$= \boxed{\frac{1}{2}\ln((x+3)^2+4) - \frac{3}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C}$$

$$= \frac{1}{2}\ln(x^2+6x+13) - \frac{3}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$$

4. (12 pts)

(a) Use Simpson's rule with $n = 4$ subdivisions to approximate the *average value* of $f(x) = e^{4x^2}$ on the interval $x = 1$ to $x = 3$. (You can leave your answer expanded out with all the correct numbers in all the correct places).

$$\text{AVE. VALUE} = \frac{1}{3-1}\int_1^3 e^{4x^2}\,dx$$

$\Delta x = \frac{3-1}{4} = \frac{1}{2}$

$x_0 = 1,\ x_1 = \frac{3}{2},\ x_2 = 2,\ x_3 = \frac{5}{2},\ x_4 = 3$

$$\approx \frac{1}{2}\cdot\frac{1}{3}\cdot\underbrace{\frac{1}{2}}_{\Delta x}\left[e^{4(1)^2} + 4e^{4(\frac{3}{2})^2} + 2e^{4(2)^2} + 4e^{4(\frac{5}{2})^2} + e^{4(3)^2}\right]$$

$$\frac{1}{12}\left[e^4 + 4e^9 + 2e^{16} + 4e^{25} + e^{36}\right] \approx 3.5929 \cdot 10^{14}$$

(b) Consider the improper integral $\int_1^4 \frac{1}{(\sqrt{x}-1)^{1/2}}\,dx$. Determine if it converges or diverges. If it converges give the value. (You MUST write as a limit, integrate and show your work).

$$\lim_{t\to 1^+}\left[\int_t^4 \frac{1}{(\sqrt{x}-1)^{1/2}}\,dx\right]$$

$u = \sqrt{x} - 1$

NEW BOUNDS $\begin{cases}\sqrt{4} - 1 = 1\\ \sqrt{t} - 1\end{cases}$

$u + 1 = \sqrt{x}$

$(u+1)^2 = x$

$2(u+1)\,du = dx$

$$\lim_{t\to 1^+}\left[\int_{\sqrt{t}-1}^{1} \frac{1}{u^{1/2}}\,2(u+1)\,du\right]$$

$$= \lim_{t\to 1^+}\left[2\int_{\sqrt{t}-1}^{1} u^{1/2} + u^{-1/2}\,du\right]$$

$$= \lim_{t\to 1^+}\left[2\left(\frac{2}{3}u^{3/2} + 2u^{1/2}\Big|_{\sqrt{t}-1}^{1}\right)\right]$$

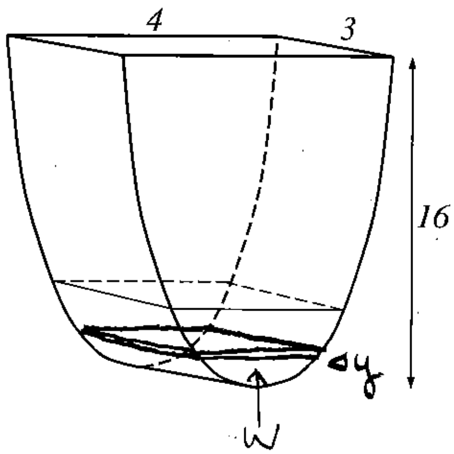
$$= \lim_{t\to 1^+} 2\left[\underbrace{\left(\frac{2}{3}+2\right)}_{8/3} - \left(\frac{2}{3}(\sqrt{t}-1)^{3/2} + 2(\sqrt{t}-1)^{1/2}\right)\right]$$

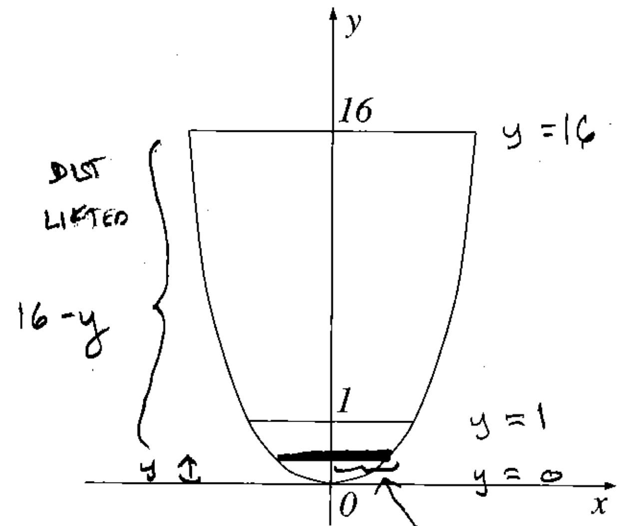
0 0

$$= \boxed{\frac{16}{3}} \approx 5.\overline{3}$$

CONVERGES

5. (12 points)



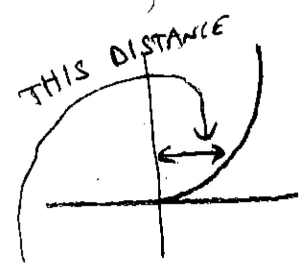


A tank is 16 feet high, with an open rectangular top of width 3 ft and length 4 ft. Each horizontal cross-section of the tank is a rectangle of fixed width 3 feet and length that changes with height. The figure above-left shows the tank. The figure above-right shows the front face of the tank, which has the shape of the function: $f(x) = 4x^2$.

Initially, there is fluid in the tank up to a height of 1 foot. The fluid weighs 15 lb/ft^3. How much work is done to empty the tank by pumping all of the fluid to the top of the tank?

FOR $0 \le y \le 1$,

$$\text{FORCE} = 15 \cdot \text{HORIZ. AREA} \cdot \Delta y$$
$$= 15 \cdot 3 \cdot W \cdot \Delta y$$



$$y = 4x^2 \Rightarrow \tfrac{1}{4} y = x^2 \Rightarrow x = \tfrac{1}{2}\sqrt{y}$$

THUS, W = TWICE THIS DIST $= 2\left(\tfrac{1}{2}\sqrt{y}\right) = \sqrt{y}$

SO FORCE $= 15 \cdot 3 \cdot \sqrt{y}\,\Delta y = 45\sqrt{y}\,\Delta y$

DIST $= 16 - y$

$$\text{WORK} = \int_0^1 (16 - y)\, 45\sqrt{y}\, dy$$
$$= 45 \int_0^1 16y^{1/2} - y^{3/2}\, dy$$
$$= 45\left[\tfrac{32}{3} y^{3/2} - \tfrac{2}{5} y^{5/2}\Big|_0^1\right]$$
$$= 45\left[\tfrac{32}{3} - \tfrac{2}{5}\right] = 480 - 18 = \boxed{462 \text{ ft-lbs}}$$