Math 124 Exam 1 Commentary

The solutions to exam 1 are on the following pages. But first read this.

Suggestions for Going Over This Exam:

1. Spend a couple minutes looking through the solutions for ALL the problems (for the ones you got right and the ones you got wrong). You might see a better way to do a problem.
2. Get out your study materials for exam 1 (the old exams you studied, the homework you studied, the notes from when you studied). Review what you studied and compare to what you missed.
3. Briefly as you look at this exam, you should see that nearly everything in this exam can be found in many places in the homework and old exams:
   (a) Page 1 and 2: Standard types of limit and deritative problems.
   (b) Page 3: Definition of derivative problem.
   (c) Page 4: Limits definitions and working from graphs.
   (d) Page 5: A typical continuity and differentiability question.
   (e) Page 6: Applied tangent line questions.

General comments about how we grade:

1. The grading is consistent! Page 1 was graded by the same TA for ALL students in my Math 124 courses. Page 2 was graded by another TA for ALL students, etc... In addition, I gave all the graders detailed grading guides. So the grading is consistent and well thought out.
2. Our job is to assess your ability to show your understanding on the test and to gauge your likelihood of success in subsequent courses. We can only grade what you put on the test. Some students try to submit a long explanation of what they 'meant' in hopes to get more credit. We don't give extra credit and I won't grade extra work that you write up now, we grade what you actually put on the test.
3. Major algebra mistakes or fundamental arthimetic mistakes (that show dramatic misunderstanding of material that preceeded this course) result in big deductions as they indicate it will be difficult for you to succeed beyond this course.
4. Major conceptual mistakes result in very big deductions as they show that you don't understand the current material.
5. We do give partial credit for showing understanding even if the final answer is wrong. However, we do not guarantee partial credit for everything you write down. For example, just copying a formula from a notesheet does not necessarily get you any credit. And we reserve the right to take off full points in certain situations where the magnitude of the mistakes outweighs anything that might be considered worth partial credit.
6. If a student makes a small calculation mistake, we typically don't take off very many points, unless that small mistake dramatically changes the problem making it much simpler than intended (in which case we must take off more points since we didn't get to see all the necessary work for the problem).
7. If it is clear that the method you show is completely wrong, but you coincidentally get an answer near the correct answer you do NOT get any points (you don't get points for pure luck).

1. (9 pts) Determine the values of the following limits or state that the limit does not exist. If it is correct to say that the limit equals ∞ or $-\infty$, then you should do so. **In all cases, show your work/reasoning. You must use algebraic methods where available. And explain in words your reasoning if an algebraic method is not available.**

(a) $\lim\limits_{x\to-3} \dfrac{\left(\frac{1}{x^2}-\frac{1}{9}\right)9x^2}{(x+3)\,9x^2} = \lim\limits_{x\to-3} \dfrac{9-x^2}{(x+3)9x^2}$

$= \lim\limits_{x\to-3} \dfrac{\cancel{(3+x)}(3-x)}{\cancel{(x+3)}9x^2}$

$= \dfrac{3-(-3)}{9(-3)^2} = \dfrac{6}{9\cdot 9} = \boxed{\dfrac{6}{81} = \dfrac{2}{27}}$

(b) $\lim\limits_{x\to\pi} \left(5+\dfrac{\cos(x)}{3\sin^2(x)}\right)$

As $x\to\pi$, the numerator $\cos(x)$ goes to -1 and the denominator $3\sin^2(x)$ goes to 0 through positive numbers.

Thus, $5+\dfrac{\cos(x)}{3\sin^2(x)}$ goes to $\boxed{-\infty}$

(c) $\lim\limits_{x\to\infty} \dfrac{6x-\sqrt{9x^2-1}}{10+5x} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{6-\sqrt{9-\frac{1}{x^2}}}{\frac{10}{x}+5}$

$= \dfrac{6-\sqrt{9-0}}{0+5} = \boxed{\dfrac{3}{5}}$

2. (9 pts) Find the indicated derivatives. (Hint: Simplify **before** you start applying derivative rules).

(a) $y = (2x^2)^3 - \frac{4x^3}{6} + \frac{xe^x}{2}$, find $\frac{dy}{dx}$.

$$y = 8x^6 - \frac{2}{3}x^3 + \frac{1}{2}xe^x$$

$$\boxed{\frac{dy}{dx} = 48x^5 - 2x^2 + \frac{1}{2}e^x + \frac{1}{2}xe^x}$$

(b) $v = \left(\sqrt{x} + \frac{1}{\sqrt[5]{x}}\right)^2$, find v'.

$$v = \left(x^{1/2} + x^{-1/5}\right)^2 = x + 2x^{\frac{1}{2}}x^{-1/5} + x^{-2/5}$$

$$\Rightarrow \quad v = x + 2x^{3/10} + x^{-2/5}$$

$$\boxed{v' = 1 + \frac{3}{5}x^{-7/10} - \frac{2}{5}x^{-7/5}}$$

(c) $f(x) = \frac{5x}{x^4} + 8x^2 \csc(x)$, find $f'(x)$.

$$f(x) = 5x^{-3} + 8x^2 \csc(x)$$

$$\boxed{f'(x) = -15x^{-4} + 16x\csc(x) - 8x^2\csc(x)\cot(x)}$$

Hint: Parts (b) and (c) can be done independently of part (a) and you can use them to check your work in part (a)!

3. (9 pts) You are keeping track of your child's height. You roughly find that the height in centimeter (cm) of your child is given by $f(t) = 60 + 30\sqrt{t}$ where t is the child's age in years.

   (a) (5 pts) Find and *completely simplify* the expression $\dfrac{f(t+h) - f(t)}{h}$.
   (Simplify until the h in the denominator cancels)

$$\frac{[60 + 30\sqrt{t+h}] - [60 + 30\sqrt{t}]}{h}$$

$$= \frac{(30\sqrt{t+h} - 30\sqrt{t})}{h} \quad \frac{(\sqrt{t+h} + \sqrt{t})}{(\sqrt{t+h} + \sqrt{t})}$$

$$= \frac{30\,(\cancel{t} + h - \cancel{t})}{h\,(\sqrt{t+h} + \sqrt{t})} = \boxed{\frac{30}{\sqrt{t+h} + \sqrt{t}}}$$

   (b) (2 pts) Find the average rate of change of the height from $t = 1$ to $t = 9$ years old.
   (Include units in your answer).

$$\frac{f(9) - f(1)}{9 - 1} = \frac{(60 + 30\sqrt{9}) - (60 + 30\sqrt{1})}{8} = \frac{60}{8} = \boxed{\frac{15}{2} = 7.5}$$

OR USE $t = 1$ AND $h = 8$ ABOVE $\quad \dfrac{30}{\sqrt{1+8} + \sqrt{1}} = \dfrac{30}{4} = \dfrac{15}{2} = 7.5 \quad \boxed{\text{cm}/\text{yr}}$

   (c) (2 pts) Find the instantaneous rate of change of height at $t = 16$ years old.
   (Include units in your answer).

$f(t) = 60 + 30\sqrt{t} = 60 + 30t^{1/2}$

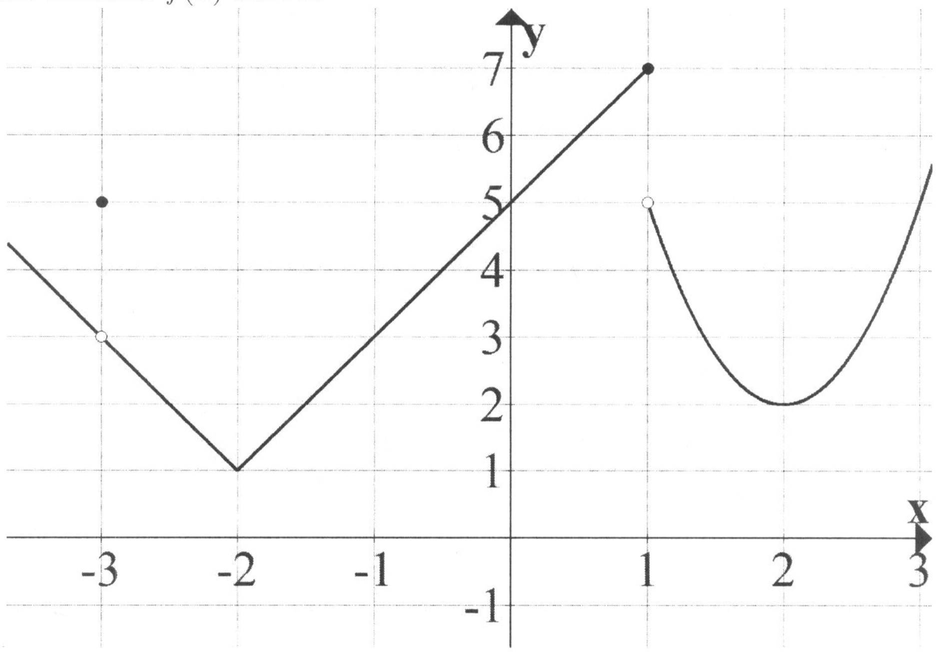
USING THE POWER RULE $\quad f'(t) = 15t^{-1/2} = \dfrac{15}{\sqrt{t}}$

SAME UNITS

OR $\quad \lim_{h \to 0} \dfrac{30}{\sqrt{t+h} + \sqrt{t}} = \dfrac{30}{2\sqrt{t}} = \dfrac{15}{\sqrt{t}}$

EITHER WAY $\quad \dfrac{15}{\sqrt{16}} = \boxed{\dfrac{15}{4} = 3.75 \quad \text{cm}/\text{yr}}$

4. Consider the function $f(x)$ shown:



(a) (2 pts) Find all solutions to $f'(x) = 0$.

THE ONLY HORIZONTAL TANGENT OCCURS AT $\boxed{X = 2}$

(b) (2 pts) Name all values of x at which $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ does not exist.

AT $\boxed{X = -3, \quad x = -2, \quad x = 1}$ THE FUNCTION IS NOT DIFFERENTIABLE

(c) (2 pts) Find $\lim_{x \to 0} \left(6f(x) + f(x-3) + \frac{\sin(2x)}{x} \right)$

$$6f(0) + \underbrace{\lim_{x \to 0} f(x-3)} + \lim_{x \to 0} 2 \frac{\sin(2x)}{2x}$$

$$6 \cdot 5 + 3 + 2 = \boxed{35}$$

(d) (3 pts) If $g(x) = x^3 f(x)$, then find $g'(-1)$.

$$g'(x) = 3x^2 f(x) + x^3 f'(x)$$

$$g'(-1) = 3(-1)^2 f(-1) + (-1)^3 \underbrace{f'(-1)}$$

$$= 3 \cdot 3 - 2$$

$$= \boxed{7}$$

"SLOPE AT" $X = -1$ IS $\frac{7-1}{1--2} = \frac{6}{3} = 2$

5. (11 pts)

(a) Let a be a constant and consider the function $f(x) = \begin{cases} \dfrac{2-\sqrt{x}}{4-x} & , \text{ if } x < 4; \\ \cos\left(\frac{\pi}{4}x\right) + \sqrt{\frac{x}{a}} & , \text{ if } x \geq 4. \end{cases}$

Find the value of a that will make $f(x)$ continuous at $x = 4$.

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{(2-\sqrt{x})}{(4-x)} \frac{(2+\sqrt{x})}{(2+\sqrt{x})} = \lim_{x \to 4^-} \frac{\cancel{(4-x)}}{\cancel{(4-x)}(2+\sqrt{x})}$$

$$= \lim_{x \to 4^-} \frac{1}{2+\sqrt{x}} = \frac{1}{2+2} = 1/4$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \left(\cos\left(\tfrac{\pi}{4}x\right) + \sqrt{\tfrac{x}{a}}\right) = \cos(\pi) + \sqrt{\tfrac{4}{a}} = -1 + \frac{2}{\sqrt{a}}$$

CONTINUOUS AT $x = 4$ $\Longleftrightarrow$ $\frac{1}{4} \stackrel{?}{=} -1 + \frac{2}{\sqrt{a}}$

$$\frac{5}{4} = \frac{2}{\sqrt{a}}$$

$$\sqrt{a} = \frac{8}{5}$$

$$\boxed{a = \frac{64}{25}}$$

(b) Let b be a constant and consider the function $g(x) = \begin{cases} bx + 10e^x & , \text{ if } x < 0; \\ 10 + \sin(x) - 4\tan(x) & , \text{ if } x \geq 0. \end{cases}$

The function $g(x)$ is continuous at $x = 0$ (you don't have to show this).

Find the value of b that makes the function differentiable at $x = 0$.

(Hint: Use your derivative shortcut rules!)

$$y = bx + 10e^x \Rightarrow \frac{dy}{dx} = b + 10e^x$$

$$y = 10 + \sin(x) - 4\tan(x) \Rightarrow \frac{dy}{dx} = \cos(x) - 4\sec^2(x)$$

As $x \to 0^-$, slope $\to b + 10e^0 = b + 10$

As $x \to 0^+$, slope $\to \cos(0) - 4\sec^2(0) = \cos(0) - 4\frac{1}{(\cos(0))^2}$

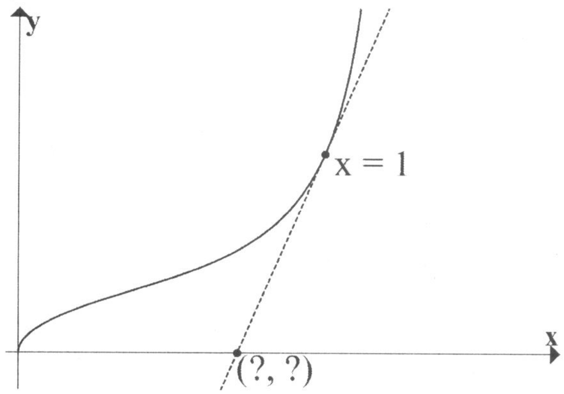
$$= 1 - 4\frac{1}{(1)^2} = -3$$

MATCHING SLOPES AT $x = 0$ $\Longleftrightarrow$ $b + 10 = -3$

$$\boxed{b = -13}$$

6. (12 pts) **NOTE: The two questions below are unrelated.**

(a) Find the tangent line to $y = \dfrac{4\sqrt{x}}{2 - x^3}$ at $x = 1$ and give the (x, y) coordinates at which this tangent line intersects the x-axis (as shown below).



$$y(1) = \frac{4\sqrt{1}}{2 - 1^3} = \frac{4}{1} = 4$$

$$y' = \frac{(2-x^3)\,2x^{-1/2} - 4\sqrt{x}\,(-3x^2)}{(2-x^3)^2}$$

$$y'(1) = \frac{(2-1)\cdot 2 - 4\sqrt{1}(-3)}{(2-1^3)^2}$$

$$= 2 + 12 = 14$$

TANGENT LINE: $y = 14(x-1) + 4$

X - INTERCEPT OF LINE: $0 = 14(x-1) + 4$
$0 = 14x - 10$

$\boxed{x = \frac{10}{14} = \frac{5}{7},\ y = 0}$

(b) Find the x and y coordinates of all points P on the curve $y = \dfrac{1}{x^2}$ at which the tangent lines at the points P have a y-intercept of 27.

$y = x^{-2}$

Let (a, b) = desired point

$\boxed{\text{I}}$ ON CURVE: $b = \dfrac{1}{a^2}$

$\boxed{\text{II}}$ TANGENT SLOPE: $y' = -2x^{-3} = -\dfrac{2}{x^3} \Rightarrow -\dfrac{2}{a^3}$

$\boxed{\text{III}}$ SLOPE FROM (a,b) TO $(0,27)$: $\dfrac{b-27}{a-0}$

OR write
$y = -\frac{2}{a^3}(x-a) + \frac{1}{a^2}$
↑ 27 ↑ 0
LEADS TO SAME EQUATION

WANT $\dfrac{b-27}{a} = -\dfrac{2}{a^3}$ AND $b = \dfrac{1}{a^2}$

$$\Rightarrow b - 27 = -\frac{2}{a^2} \Rightarrow \frac{1}{a^2} - 27 = -\frac{2}{a^2}$$

$$\Rightarrow \frac{3}{a^2} = 27 \Rightarrow \frac{1}{9} = a^2 \qquad a = \pm\sqrt{\tfrac{1}{9}} = \pm\tfrac{1}{3}$$

$$b = \frac{1}{(\pm 1/3)^2} = 9$$

$\boxed{(x,y) = \left(-\tfrac{1}{3}, 9\right) \text{ or } \left(\tfrac{1}{3}, 9\right)}$