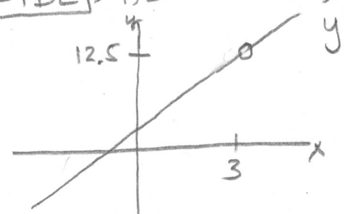
1. (16 pts) Determine the values of the following limits or state that the limit does not exist. If it is correct to say that the limit equals ∞ or $-\infty$, then you should do so. In all cases, show your work/reasoning.

(a) $\displaystyle\lim_{x\to 3} \frac{5x^2 - 5x - 30}{2x - 6}$ $\displaystyle = \lim_{x\to 3} \frac{5(x^2 - x - 6)}{2(x-3)} = \lim_{x\to 3} \frac{5\cancel{(x-3)}(x+2)}{2\cancel{(x-3)}}$

$\displaystyle = \frac{5(3+2)}{2}$

$\displaystyle = \boxed{\frac{25}{2} = 12.5}$

ASIDE: HERE'S THE GRAPH $y = \frac{5x^2-5x-30}{2x-6}$ which is the same as $y = \frac{5(x+2)}{2}$ except at $x = 3$.



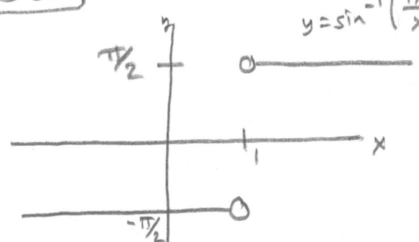
(b) $\displaystyle\lim_{x\to 1^-} \sin^{-1}\left(\frac{|x-1|}{x-1}\right)$ $\displaystyle = \sin^{-1}\left(\lim_{x\to 1^-} \frac{|x-1|}{x-1}\right) = \sin^{-1}\left(\lim_{x\to 1^-} \frac{-(x-1)}{(x-1)}\right)$

$= \sin^{-1}(-1)$

$= \boxed{-\pi/2}$

NOTE: YOU SHOULD ALWAYS GIVE YOUR ANSWER IN RADIANS UNLESS OTHERWISE SPECIFIED

ASIDE: HERE'S THE GRAPH $y = \sin^{-1}\left(\frac{|x-1|}{x-1}\right)$ BECAUSE $\frac{|x-1|}{x-1} = \begin{cases} -1, & x < 1; \\ 1, & x > 1. \end{cases}$



(c) $\displaystyle\lim_{x\to\infty} \left(\sqrt{3x + 4x^2 - 1} - 2x\right)$ $\displaystyle \cdot \frac{\sqrt{3x+4x^2-1}+2x}{\sqrt{3x+4x^2-1}+2x} = \lim_{x\to\infty} \frac{3x + \cancel{4x^2} - 1 - \cancel{4x^2}}{(\sqrt{3x+4x^2-1}+2x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

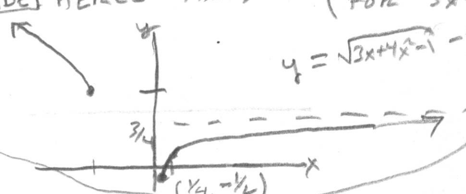
NOTE: $\infty - \infty$ IS NOT ZERO IN GENERAL. THIS IS AN INDETERMINANT FORM.

$\displaystyle = \lim_{x\to\infty} \frac{3 - \frac{1}{x}}{\sqrt{\frac{3}{x} + 4 - \frac{1}{x^2}} + 2} = \frac{3-0}{\sqrt{0+4-0}+2}$

$\frac{1}{x} = \sqrt{\frac{1}{x^2}}$ because $x > 0$

$= \boxed{\frac{3}{4}}$

ASIDE: HERE'S THE GRAPH (ONLY DEFINED FOR $3x+4x^2-1 \geq 0$)



(d) $\displaystyle\lim_{x\to 5^+} \frac{1 - x^2}{\sqrt{x - 5}}$

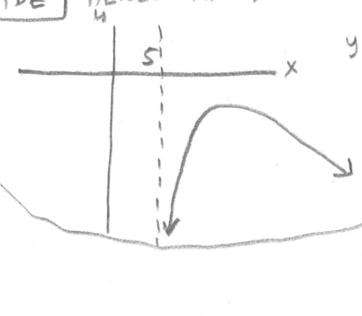
AS $x \to 5^+$, WE HAVE $\sqrt{x-5} \to 0^+$

AND $1 - x^2 \to -24$

FOR x SLIGHTLY LARGER THAN 5, $\frac{1-x^2}{\sqrt{x-5}} = \frac{\sim -24}{\text{small positive}} = \text{large negative}$.

HENCE $\displaystyle\lim_{x\to 5^+} \frac{1-x^2}{\sqrt{x-5}} = \boxed{-\infty}$

ASIDE HERE'S THE GRAPH ($x > 5$) $y = \frac{1-x^2}{\sqrt{x-5}}$



2. (5 pts) Determine the value of the limit (explain your work): $\lim_{x\to-\infty} \dfrac{(2+e^x)x^2+3\cos(x)}{(x^2\tan^{-1}(x)-4\pi)}$ $\dfrac{1/x^2}{1/x^2}$

$$\lim_{x\to-\infty} \frac{(2+e^x)+\frac{3\cos(x)}{x^2}}{\tan^{-1}(x)-\frac{4\pi}{x^2}}$$

$$= \frac{2+0+0}{-\pi/2 - 0} = \boxed{-4/\pi}$$

① $\lim_{x\to-\infty} e^x = 0$

RADIANS

② $\lim_{x\to-\infty} \tan^{-1}(x) = -\pi/2$

$-\pi/2$

③ $\lim_{x\to-\infty} \frac{4\pi}{x^2} = 0$

④ SQUEEZE THM

$-1 \le \cos(x) \le 1$

$\Rightarrow -\frac{3}{x^2} \le \frac{3\cos(x)}{x^2} \le \frac{3}{x^2}$

AND $\lim_{x\to-\infty} -\frac{3}{x^2} = 0 = \lim_{x\to-\infty} \frac{3}{x^2}$

THUS, $\lim_{x\to-\infty} \frac{3\cos(x)}{x^2} = 0$

3. (7 pts) Let $f(x) = e^{\sin(2x)} + \tan(x)$. Find $f'(0)$ and $f''(0)$.

CHAIN RULE TWICE

$$f'(x) = \underbrace{2\cos(2x)}_{F}\,\underbrace{e^{\sin(2x)}}_{S} + \sec^2(x)$$

PRODUCT RULE

CHAIN RULE

$$f'(0) = 2\cos(0)e^0 + \sec^2(0) = 2+1 = \boxed{3}$$

$$f''(x) = 2\cos(2x)\,2\cos(2x)\,e^{\sin(2x)} - 4\sin(2x)e^{\sin(2x)} + 2\sec(x)\sec(x)\tan(x)$$

$$f''(x) = (4\cos^2(2x) - 4\sin(2x))e^{\sin(2x)} + 2\sec^2(x)\tan(x)$$

$$f''(0) = (4\cdot1^2 - 4\cdot0)\cdot1 + 2\cdot1^2\cdot0 = \boxed{4}$$

4. (8 pts) Let $f(x) = \dfrac{7x+1}{\sqrt{x^2+3}}$.

(a) Find the equations of the tangent line and normal line to $f(x)$ at the point on the graph where $x = 1$.

$$f'(x) = \frac{\sqrt{x^2+3}\cdot 7 - (7x+1)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}}}{x^2+3} \cdot \frac{\sqrt{x^2+3}}{\sqrt{x^2+3}} = \frac{7(x^2+3) - (7x+1)x}{(x^2+3)^{3/2}}$$

$$= \frac{7x^2+21-7x^2-x}{(x^2+3)^{3/2}} = \frac{21-x}{(x^2+3)^{3/2}}$$

$$f(1) = \frac{7(1)+1}{\sqrt{(1)^2+3}} = \frac{8}{2} = 4 \qquad f'(1) = \frac{21-1}{(1^2+3)^{3/2}} = \frac{20}{8} = \frac{5}{2} = 2.5$$

TANGENT: $y = \frac{5}{2}(x-1)+4$

NORMAL: $y = -\frac{2}{5}(x-1)+4$

(b) Find the x-coordinates of all points on the graph of $f(x)$ at which the tangent line is horizontal.

HORIZONTAL TANGENT $\Leftrightarrow$ $f'(x) = 0$

$$\frac{21-x}{(x^2+3)^{3/2}} = 0$$

$$21 - x = 0$$

$$\boxed{x = 21}$$

5. (4 pts) Suppose $f(x)$ is some function that is defined for all real numbers (you don't know if $f(x)$ is even, odd or neither). For each new function defined below, determine if the function **must** be even, odd, or neither:

(a) $g(x) = f(x) - f(-x)$ CIRCLE ONE: EVEN ODD NEITHER

(b) $h(x) = f(x) + f(-x)$ CIRCLE ONE: EVEN ODD NEITHER

(c) $k(x) = |f(x)|$ CIRCLE ONE: EVEN ODD NEITHER

(d) $j(x) = f(x)f(-x)$ CIRCLE ONE: EVEN ODD NEITHER

a $g(-x) = f(-x) - f(x) = -(f(x) - f(-x)) = -g(x) \Rightarrow$ ODD

b $h(-x) = f(-x) + f(x) = f(x) + f(-x) = g(x) \Rightarrow$ EVEN

c $k(-x) = |f(-x)|$ CAN'T DETERMINE RELATIONSHIP WITH $k(x) = |f(x)| \Rightarrow$ NEITHER

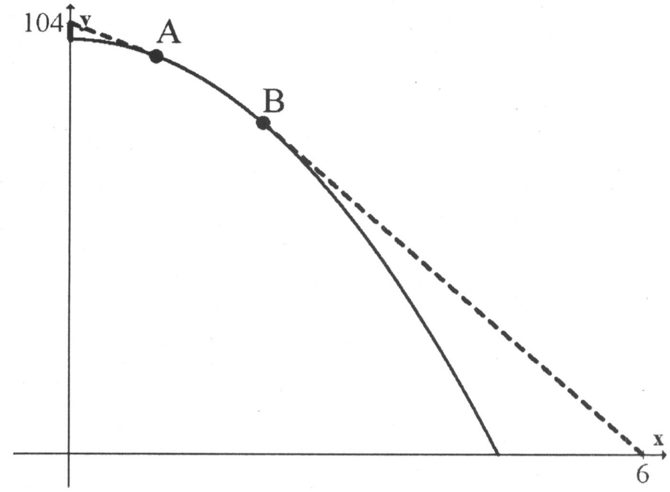
d $j(-x) = f(-x)f(x) = f(x)f(-x) = j(x) \Rightarrow$ EVEN

6. (10 points)

The side view of a hill, depicted to the right, is given by the function

$$y = -5x^2 + 100,$$

where x and y are in feet.
A long rope is tied to the ground at the point (6,0) and to the top of a 4 foot post at the very top of the hill. The rope is pulled tight so that it bends to form two separate straight line segments adjoining a segment of rope in the middle that follows the curve of the hill.



Find the points A and B at which the rope first touches the hill on each side. (Hint: The straight lines form two different tangent lines to the hill).

Let $A = (a,b) = (a, -5a^2+100)$
$B = (c,d) = (c, -5c^2+100)$

SLOPE OF TANGENT $= y' = -10x$

$\boxed{A}$ WANT SLOPE FROM (a,b) TO $(0,104)$ TO EQUAL THE SLOPE OF THE TANGENT AT a

$$-10a = \frac{b-104}{a-0} = \frac{-5a^2+100-104}{a}$$

$$-10a^2 = -5a^2 - 4$$

$$4 = 5a^2$$

$$a^2 = 4/5 \qquad a = \pm\sqrt{4/5} = \frac{2}{\sqrt{5}}$$

$$b = -5a^2+100 = -5(4/5)+100 = 96$$

$$\boxed{A = \left(\frac{2}{\sqrt{5}}, 96\right)}$$

$\boxed{B}$ WANT SLOPE FROM (c,d) TO $(6,0)$ TO EQUAL THE SLOPE OF THE TANGENT AT c

$$-10c = \frac{d-0}{c-6} = \frac{-5c^2+100}{c-6}$$

$$-10c^2+60c = -5c^2+100$$

$$0 = 5c^2 - 60c + 100$$

$$0 = c^2 - 12c + 20$$

$$0 = (c-2)(c-10)$$

$$c = 2, \quad \cancel{c = 10}$$

$$d = -5(2)^2+100 = 80$$

$$\boxed{B = (2, 80)}$$

7. (10 pts) Determine the values for the constants a and b such that the following function, f, is **continuous** and **differentiable** for all real numbers x.

$$f(x) = \begin{cases} ax^2 + bx - 1 & \text{, if } x \le 1; \\ \dfrac{a+bx}{x+1} & \text{, if } x > 1. \end{cases}$$

NOTE: Separately, continuous and differential on the given domains

$\boxed{\text{CONTINUITY}}$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$$a+b-1 = \frac{a+b}{1+1} \Rightarrow 2a+2b-2 = a+b$$

$$\boxed{a+b=2}$$

NOTE: SINCE WE WILL FORCE CONTINUITY, BOTH FORMULAS WILL GIVE SAME f(1) VALUE

$\boxed{\text{DIFFERENTIABILITY}}$ $\underbrace{\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h}}$ $= \lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h}$

$$\frac{d}{dx}\left[ax^2+bx-1\right]\Big|_{x=1} = \frac{d}{dx}\left[\frac{a+bx}{x+1}\right]\Big|_{x=1}$$

$$(2ax+b)\Big|_{x=1} = \left(\frac{(x+1)b - (a+bx)}{(x+1)^2}\right)\Big|_{x=1}$$

$$2a+b = \frac{2b-(a+b)}{2^2}$$

$$8a+4b = b-a$$

$$\boxed{9a+3b=0} \qquad 9a = -3b$$

$$b = -3a$$

$\boxed{\text{COMBINE}}$ $b=-3a$ $\qquad a+b=2$

$$a - 3a = 2$$

$$-2a = 2 \qquad a = -1$$

$$b = -3(-1) = 3$$

$$\boxed{\begin{array}{l} a=-1 \\ b=3 \end{array}}$$

ASIDE: HERE'S THE GRAPH

$y = -x^2+3x-1$

$y' = -2x+3$

$y'(1) = 1$

$y = \dfrac{-1+3x}{x+1}$

$y' = \dfrac{4}{(x+1)^2}$

$y'(1) = 1$