1. (12 pts) Compute the slopes of the tangents at the specified points. Give your final answers simplified in exact form.

(a) Let $f(x) = \ln(\tan^{-1}(x^3))$. Find the slope of the tangent line at $x = 1$.

$$f'(x) = \frac{1}{\tan^{-1}(x^3)} \frac{1}{1+(x^3)^2} \cdot 3x^2$$

$$f'(1) = \frac{1}{\tan^{-1}((1)^3)} \frac{1}{1+(1)^6} 3\cdot(1)^2 = \frac{3}{2} \frac{1}{\tan^{-1}(1)}$$

$$= \frac{3}{2} \frac{1}{\pi/4} = \boxed{\frac{6}{\pi}}$$

(b) Let $y = (e^{2x} - \ln(x))^{10}$. Find the slope of the tangent at $x = 1$.

$$y' = 10(e^{2x} - \ln(x))^9 (2e^{2x} - \tfrac{1}{x})$$

$$y'(1) = 10(e^{2(1)} - \ln(1))^9 (2e^{2(1)} - \tfrac{1}{1})$$

$$= 10(e^2)^9(2e^2 - 1)$$

$$= \boxed{10e^{18}(2e^2 - 1) = 20e^{20} - 10e^{18}}$$

(c) Let $y = \frac{1}{2}(x)^{\sqrt{x}}$. Find the slope of the tangent line at $x = 4$.

$$2y = x^{\sqrt{x}}$$

$$\frac{d}{dx}\left[\ln(2y) = \sqrt{x}\ln(x)\right]$$

$$\frac{1}{\cancel{2}y} \cancel{2} \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln(x) + \frac{\sqrt{x}}{x} \quad \frac{1}{\sqrt{x}}$$

$$\frac{dy}{dx} = y\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

$$y(4) = \tfrac{1}{2} 4^{\sqrt{4}} = \tfrac{1}{2}16 = 8$$

$$\frac{dy}{dx}(4) = 8\left(\frac{\ln(4)}{2\sqrt{4}} + \frac{1}{\sqrt{4}}\right)$$

$$= 8\left(\frac{\ln(4)}{4} + \frac{1}{2}\right)$$

$$= \boxed{2\ln(4) + 4}$$

$$= \boxed{4\ln(2) + 4}$$

$$= \boxed{4(\ln(2) + 1)}$$

2. (7 pts) Consider the implicitly defined curve given by: $x^2 + y^2 = \cos(x) + 3xy^3$.
This curve has two y-intercepts. Find the **equations of the tangent lines** at each y-intercept.

$$x = 0 \Rightarrow 0^2 + y^2 = \cos(0) + 3(0)y^3 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

y-intercepts: $(0,-1)$ and $(0,1)$

SLOPE: $$2x + 2y\,y' = -\sin(x) + 3y^3 + 9xy^2 y'$$

$$y' = \frac{3y^3 - \sin(x) - 2x}{2y - 9xy^2}$$

AT $(0,-1)$, $y' = \dfrac{3(-1)^3 - \sin(0) - 2(0)}{2(-1) - 9(0)(-1)^2} = \dfrac{3}{2}$

AT $(0,1)$, $y' = \dfrac{3(1)^3 - \sin(0) - 2(0)}{2(1) - 9(0)(1^2)} = \dfrac{3}{2}$

$$\boxed{y = \tfrac{3}{2}x - 1 \qquad y = \tfrac{3}{2}x + 1}$$

3. (7 pts) Find the absolute maximum and minimum values of $f(x) = x^2 - 3\ln(x^2+1)$ on the interval $[-1, 3]$. Justify your answer.

CRITICAL NUMBERS: $$f'(x) = 2x - 3\frac{2x}{x^2+1} = 2x - \frac{6x}{x^2+1}$$

$$2x - \frac{6x}{x^2+1} \overset{?}{=} 0 \Rightarrow 2x(x^2+1) - 6x = 0$$

$$2x^3 + 2x - 6x = 0$$

$$2x^3 - 4x = 0$$

$$2x(x^2-2) = 0$$

$$\underbrace{x = -\sqrt{2}}_{\text{NOT IN SPECIFIED DOMAIN}}, \quad x = 0, \quad x = \sqrt{2}$$

EVALUATE:

$f(-1) = (-1)^2 - 3\ln((-1)^2+1) = 1 - 3\ln(2) \approx -1.079441542$

$f(0) = 0^2 - 3\ln(0^2+1) = 0$

ABS MIN $\rightarrow$ $f(\sqrt{2}) = \sqrt{2}^2 - 3\ln(\sqrt{2}^2+1) = \boxed{2 - 3\ln(3)} \approx -1.295836866$

ABS MAX $\rightarrow$ $f(3) = 3^2 - 3\ln(3^2+1) = \boxed{9 - 3\ln(10)} \approx 2.092244721$

4. (10 pts) Suppose we want to approximate the solution(s) to $\cos(x) = x$. We let $f(x) = \cos(x) - x$.

(a) Precisely explain in words why $f(x) = 0$ has exactly one solution between $x = 0$ and $x = \pi$. Your explanation should contain 2-3 sentences. As part of your explanation, find $f'(x)$.

Since $\boxed{\begin{array}{l} f(0) = \cos(0) - 0 = 1 > 0 \\ f(\pi) = \cos(\pi) - \pi = -1 - \pi < 0 \end{array}}$ and
and
f is continuous, there must be at least one sol'n to $f(x) = 0$ between $x=0$ and $x = \pi$. (Intermediate Value theorem)

Since $\boxed{f'(x) = -\sin(x) - 1}$ $f'(x)$ is negative for all x between 0 and π. Thus, $f(x)$ is always $\boxed{\text{decreasing}}$ between $x=0$ and $x=\pi$. So there is exactly on sol'n between $x=0$ and $x=\pi$.

(b) Through experimentation, you find the solution is near $x = \frac{\pi}{4}$. Find the linear approximation to $f(x) = \cos(x) - x$ at $x = \frac{\pi}{4}$ and use it to approximate the solution to $f(x) = 0$. (In other words, do the first step of Newton's method).

$$f\left(\frac{\pi}{4}\right) = \cos(\pi/4) - \pi/4 = \frac{\sqrt{2}}{2} - \pi/4$$

$$f'(\pi/4) = -\sin(\pi/4) - 1 = -\frac{\sqrt{2}}{2} - 1$$

$$L(x) = \left(-\frac{\sqrt{2}}{2} - 1\right)(x - \pi/4) + \frac{\sqrt{2}}{2} - \pi/4 \approx \cos(x) - x$$

for $x \approx \pi/4$.

$$\cos(x) - x = 0 \Leftrightarrow \left(-\frac{\sqrt{2}}{2} - 1\right)(x - \pi/4) + \frac{\sqrt{2}}{2} - \pi/4 \approx 0$$

$$x - \pi/4 = \frac{-\sqrt{2}/2 + \pi/4}{-\sqrt{2}/2 - 1}$$

$$x = \pi/4 - \frac{(-\sqrt{2}/2 + \pi/4)}{\sqrt{2}/2 + 1} = \pi/4 + \frac{2\sqrt{2} - \pi}{2\sqrt{2} + 4}$$

$$\boxed{x \approx 0.739561335}$$

ASIDE

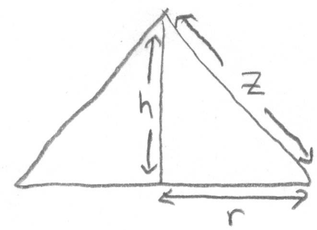
ACTUAL VALUE $x = 0.7390851 3$

5. (12 pts) Sand is being dumped from a conveyor belt at a rate of 12 [redacted]π ft^3/min, forming a conical pile of sand. Four minutes later, the radius of the base of the pile is 6 feet and the height is [redacted] 4 feet, and the radius is increasing at a rate of [redacted] ft/min. 1/4
(Recall: The volume of a cone is $V = \frac{\pi}{3}r^2h$).

(a) How fast is the height changing four minutes after the pile starts?

$$V = \frac{\pi}{3} r^2 h$$

$$\Rightarrow \frac{dV}{dt} = \frac{2\pi}{3} r h \frac{dr}{dt} + \frac{\pi}{3} r^2 \frac{dh}{dt}$$



GIVEN: $r = 6$, $h = 4$, $\frac{dr}{dt} = 1$, $\frac{dV}{dt} = 12\pi$

So

$$12\pi = \underbrace{\frac{2\pi}{3}(6)(4)\tfrac{1}{4}}_{4\pi} + \underbrace{\frac{\pi}{3}(6)^2}_{12\pi} \frac{dh}{dt}$$

$$12\pi - 4\pi = 12\pi \frac{dh}{dt}$$

$$8 = 12 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{8}{12} = \boxed{\frac{2}{3} = 0.\overline{6} \text{ ft/min}}$$

(b) Viewed from the side, the conical pile is an isosceles triangle with base of length $2r$ and the other two sides of equal length. How fast are the other side lengths of the triangle changing four minutes after the pile starts?

$$z^2 = r^2 + h^2$$

$$\cancel{2} z \frac{dz}{dt} = \cancel{2} r \frac{dr}{dt} + \cancel{2} h \frac{dh}{dt}$$

GIVEN: $r = 6$, $h = 4$, $\frac{dr}{dt} = \frac{1}{4}$, $\frac{dh}{dt} = \frac{2}{3}$,

$$z = \sqrt{r^2 + h^2} = \sqrt{6^2 + 4^2}$$

$$z = \sqrt{52}$$

$$\sqrt{52} \frac{dz}{dt} = 6\left(\frac{1}{4}\right) + 4 \cdot \left(\frac{2}{3}\right)$$

$$\sqrt{52} \frac{dz}{dt} = \frac{3}{2} + \frac{8}{3} = \frac{9 + 16}{6} = \frac{25}{6}$$

$$\frac{dz}{dt} = \boxed{\frac{25}{6\sqrt{52}} \approx 0.5778127044 \text{ ft/min}}$$

6. (12 pts) You are on a ferris wheel. Let the origin be the location where the ferris wheel is attached to the ground. You start keeping track of your location (at $t = 0$ seconds) and you find that as the wheel rotates, your coordinates (in feet) are given by the parametric equations

$$x(t) = 70\cos\left(\frac{\pi}{15}t\right)$$
$$y(t) = 70\sin\left(\frac{\pi}{15}t\right) + 74$$

(a) (3 pts) Give the following information:

Radius of the Wheel = 70 ft

Initial, $t = 0$, location (x, y) = $(70, 74)$

Time it takes to complete one revolution = 30 seconds

(b) (5 pts) Find the slope of the tangent at $t = 10$ seconds.

$x'(t) = -\frac{70\pi}{15}\sin\left(\frac{\pi}{15}t\right)$ $\quad$ $x'(10) = -\frac{70\pi}{15}\cdot\frac{\sqrt{3}}{2}$

$y'(t) = \frac{70\pi}{15}\cos\left(\frac{\pi}{15}t\right)$ $\quad$ $y'(10) = \frac{70\pi}{15}\cdot\left(-\frac{1}{2}\right)$

$$\text{SLOPE} = \frac{y'(10)}{x'(10)} = \frac{-\cancel{\frac{70\pi}{15}}\cdot\frac{1}{2}}{-\cancel{\frac{70\pi}{15}}\cdot\frac{\sqrt{3}}{2}} = \boxed{\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}}$$

(c) (4 pts) At the instant when you reach the far left point on the ferris wheel, you 'accidentally' throw a water balloon directly downward. At $t = 1$ seconds after the instant you threw the balloon, it lands on Dr. Loveless' head who happens to be standing on the ground below. Dr. Loveless is 6 feet tall.

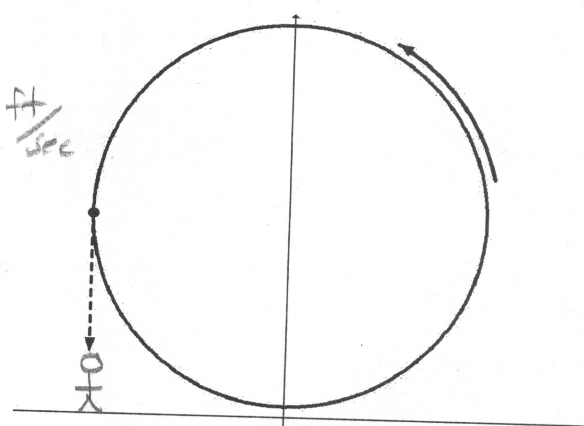
The balloon's height, t, seconds after being thrown is given by $h(t) = 74 - At - 16t^2$, where A is the magnitude of the initial vertical velocity of the balloon (the vertical speed of the ferris wheel plus the speed at which your arm threw the balloon).

At what initial speed did your arm throw the balloon?

$h(1) = 6 \Rightarrow 74 - A - 16 = 6$

$A = 74 - 16 - 6 = 52$ ft/sec

initial speed of balloon



vertical velocity of ferris wheel at at this point ($t = 15$ seconds above)

$y'(15) = \frac{70\pi}{15}\cos\left(\frac{\pi}{15}15\right) = -\frac{70\pi}{15}$

VERT. SPEED OF FERRIS WHEEL $= \frac{70\pi}{15} \approx 14.66076572$ ft/sec

VERT. SPEED OF THROWING $= \boxed{52 - \frac{70\pi}{15} \approx 37.33923428 \text{ ft/sec}}$