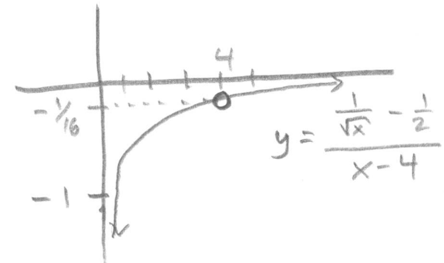
1. (16 pts) Determine the values of the following limits or state that the limit does not exist. If it is correct to say that the limit equals ∞ or $-\infty$, then you should do so. **In all cases, show your work/reasoning.**

(a) $\displaystyle\lim_{x\to 4}\frac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}\cdot\frac{2\sqrt{x}}{2\sqrt{x}} = \lim_{x\to 4}\frac{(2-\sqrt{x})}{2\sqrt{x}(x-4)}\frac{(2+\sqrt{x})}{(2+\sqrt{x})}$

$\displaystyle = \lim_{x\to 4}\frac{4-x}{2\sqrt{x}(x-4)(2+\sqrt{x})}$

$\displaystyle = \lim_{x\to 4}\frac{-\cancel{(x-4)}}{2\sqrt{x}\cancel{(x-4)}(2+\sqrt{x})}$

$\displaystyle = \frac{-1}{2\sqrt{4}(2+\sqrt{4})} = \boxed{-\frac{1}{16}}$

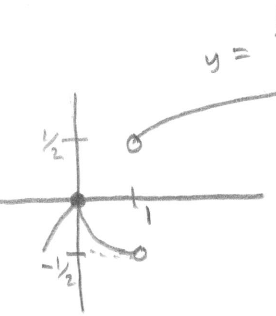


(b) $\displaystyle\lim_{x\to 1^-}\frac{|x^2-x|}{x^2-1} = \lim_{x\to 1^-}\frac{|x(x-1)|}{(x-1)(x+1)}$

For $0<x<1$, $x(x-1)<0$, So $|x(x-1)|=-x(x-1)$

$\displaystyle = \lim_{x\to 1^-}\frac{-x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}$

$\displaystyle = -\frac{1}{1+1} = \boxed{-\frac{1}{2}}$

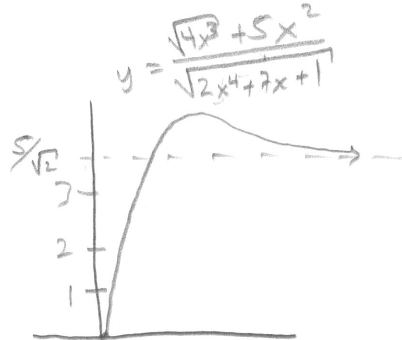


(c) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{4x^3}+5x^2}{\sqrt{2x^4+7x+1}}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

NOTE: $x^2=\sqrt{x^4}$

$\displaystyle = \lim_{x\to\infty}\frac{\sqrt{\frac{4x^3}{x^4}}+5}{\sqrt{\frac{2x^4}{x^4}+\frac{7x}{x^4}+\frac{1}{x^4}}} = \frac{\sqrt{0}+5}{\sqrt{2+0+0}} = \boxed{\frac{5}{\sqrt{2}}} = \frac{5}{2}\sqrt{2}$

≈ 3.535533906



(d) $\displaystyle\lim_{x\to 0}\frac{\sin(x)-\tan^{-1}\left(\frac{1}{x^2}\right)+7}{x}$ $\boxed{\text{DNE}}$

Note: $\displaystyle\lim_{x\to 0}\sin(x)=0$

$\displaystyle\lim_{x\to 0}\tan^{-1}\left(\frac{1}{x^2}\right)=\pi/2$

The numerator is approaching $0-\pi/2+7$, which is a nonzero (positive) constant.

The denominator is approaching 0.

Thus, $\displaystyle\lim_{x\to 0^-}\frac{\sin(x)-\tan^{-1}\left(\frac{1}{x^2}\right)+7}{x}=-\infty$

$\displaystyle\lim_{x\to 0^+}\frac{\sin(x)-\tan^{-1}\left(\frac{1}{x^2}\right)+7}{x}=\infty$

2. (7 pts) Let $f(x) = 12\sqrt[3]{x} + \dfrac{2x}{x^6} - x^5 e^{3x}$. Find the equation for the tangent line to $f(x)$ at $x = 1$.

$$f(x) = 12x^{1/3} + 2x^{-5} - x^5 e^{3x}$$

$$f'(x) = 4x^{-2/3} - 10x^{-6} - 5x^4 e^{3x} - 3x^5 e^{3x}$$

$$f'(1) = 4 - 10 - 5e^3 - 3e^3 = -6 - 8e^3.$$

$$f(1) = 12 + 2 - e^3 = 14 - e^3.$$

$$\boxed{y = (-6-8e^3)(x-1) + (14-e^3)}$$

3. (7 pts) Let $g(x) = \dfrac{1}{x+1}$. Find a number $x = a$, greater than 1, such that the slope of the **secant** line to $g(x)$ from $x = 1$ to $x = a$ is exactly $-\frac{1}{24}$.

$$\frac{g(a) - g(1)}{a - 1} = -\frac{1}{24}$$

$$\frac{\frac{1}{a+1} - \frac{1}{1+1}}{a-1} = -\frac{1}{24}$$

$$\frac{1}{a+1} - \frac{1}{2} = -\frac{1}{24}a + \frac{1}{24}$$

$$\frac{1}{a+1} = -\frac{1}{24}a + \frac{13}{24}$$

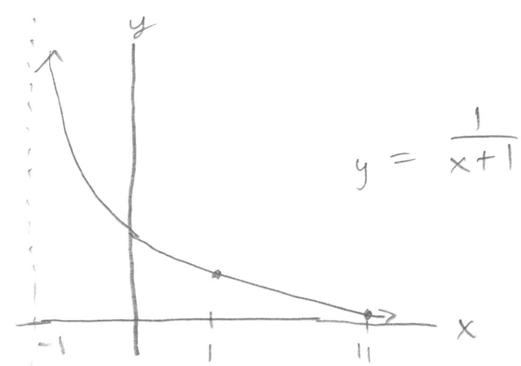
$$24 = (-a+13)(a+1)$$

$$24 = -a^2 + 12a + 13$$

$$a^2 - 12a + 11 = 0$$

$$(a-1)(a-11) = 0$$

$$\boxed{a = 11}$$



3. (10 pts) To get back at your instructor for a challenging exam you make him take a ride on a giant slingshot at a nearby carnival. Your instructor is attached to two giant bungee cords that launch him vertical straight up, then he oscillates up and down.
Suppose your instructor's height off the ground after, t, seconds is given by

$$y(t) = \frac{50e^{2t} - 40\cos(2t)}{e^{2t}} \quad \text{feet.}$$

(a) (3 pts) Find $\lim_{t\to\infty} y(t)$. (Justify your answer).

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 50 - 40e^{-2t}\cos(2t)$$
$$= 50 - 40\cdot 0$$
$$= \boxed{50} \text{ ft}$$

Since $-1 \le \cos(2t) \le 1$

$-e^{-2t} \le e^{-2t}\cos(2t) \le e^{-2t}$

and as $t \to \infty$, $e^{-2t} \to 0$,

thus $e^{-2t}\cos(2t) \to 0$

(b) (4 pts) Find your instructor's initial velocity. PRODUCT RULE AND CHAIN RULE

$$y'(t) = 80e^{-2t}\cos(2t) + 80e^{-2t}\sin(2t)$$
$$y'(0) = 80\cdot 1\cdot 1 + 0 = \boxed{80 \text{ ft/sec}}$$

(c) (3 pts) Find the first positive time when your instructor's velocity is zero.

$$80e^{-2t}\cos(2t) + 80e^{-2t}\sin(2t) \overset{?}{=} 0$$
$$80e^{-2t}(\cos(2t) + \sin(2t)) = 0$$
$$\cos(2t) + \sin(2t) = 0$$
$$\left.\begin{array}{l}\sin(2t) = -\cos(2t)\\ \tan(2t) = -1\end{array}\right\} \quad \begin{array}{l} 2t = \ldots, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots \\ t = \ldots, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \ldots \end{array}$$

First positive time $\boxed{t = 3\pi/8 \text{ seconds}}$

~~5~~4. (10 pts) Let a and b be constants. Consider the function $f(x) = \begin{cases} x|x+2| & \text{, if } x < -1; \\ x^{1/3} & \text{, if } -1 \le x \le 1; \\ ax+b & \text{, if } x > 1. \end{cases}$

(a) Find a and b so that $f(x)$ is continuous and differentiable at $x = 1$.

For continuity, we need $\underbrace{\lim_{x\to 1^-} f(x)}_{1^{1/3}} = \underbrace{\lim_{x\to 1^+} f(x)}_{a(1)+b} = f(1) = 1^{1/3}$

$1^{1/3} = a(1) + b$

Thus, $a + b = 1$

For differentiability, we need $\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \frac{1}{3}x^{-2/3}\Big|_{x=1} = 1/3$

to equal $\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{a(1+h)+b-1}{h}$

$= \lim_{h\to 0^+} \frac{a+ah+b-1}{h}$ $a+b-1=0$

$= a$

Thus, $a = 1/3$

$\Rightarrow b = 2/3$

$$\boxed{a = \frac{1}{3}, \quad b = \frac{2}{3}}$$

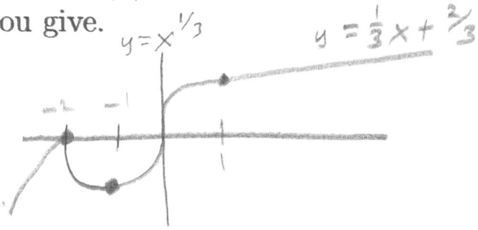
(b) Use your values ~~of~~ for a and b from part (a), $f(x)$ is now continuous everywhere. However, it is NOT differentiable everywhere. Find all values of x at which $f(x)$ is not differentiable. Justify why the function is not differentiable at the values you give.

NOTE: FOR $x < -2$, $x|x+2| = -x(x+2)$

$y = -x^2 - 2x$

for $x \ge -2$, $x|x+2| = x(x+2)$

$y = x^2 + 2x$



NOT DIFFERENTIABLE AT $\boxed{x = -2, \; x = -1, \text{ and } x = 0}$

At $x = -2$, $\lim_{x\to -2^-} f'(x) = \lim_{x\to -2^-} -2x - 2 = 2$

$\lim_{x\to -2^+} f'(x) = \lim_{x\to -2^+} 2x + 2 = -2$

SLOPES NOT EQUAL.

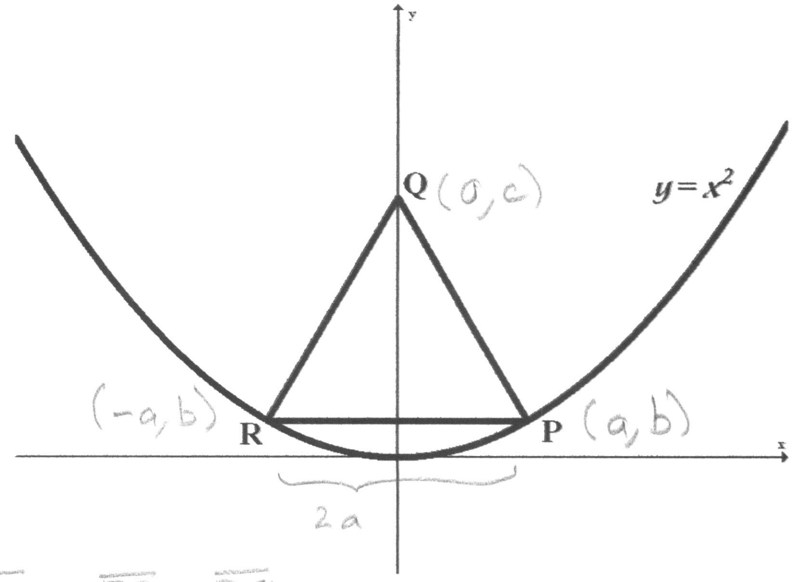
At $x = -1$, $\lim_{x\to -1^-} f'(x) = \lim_{x\to -1^-} 2x+2 = 0$

$\lim_{x\to -1^+} f'(x) = \lim_{x\to -1^+} \frac{1}{3}x^{-2/3} = 1/3$

SLOPES NOT EQUAL

At $x = 0$, vertical tangent.

6. (10 pts) Consider the parabola $y = x^2$ in the figure below. The triangle is an **equilateral** triangle and the segments $\overline{PQ}$ and $\overline{RP}$ are **normal** (*i.e* perpendicular) to the parabola at P and R, respectively. Find the coordinates of Q.



$y' = 2x$

① Equilateral Triangle $\Rightarrow \overline{PQ} = \overline{RP} = \overline{RQ}$

So $\sqrt{a^2 + (b-c)^2} = 2a \Rightarrow a^2 + (b-c)^2 = 4a^2$

$(b-c)^2 = 3a^2$

② On parabola $\Rightarrow$ $b = a^2$ $\qquad b - c = \pm\sqrt{3}a$

$b - c > 0$ ✓

$a^2 - c = \sqrt{3}a$

So $c = a^2 + \sqrt{3}a$.

③ Slope of tangent at $P = 2a$

slope of normal at $P = -\frac{1}{2a}$

NORMAL LINE: $y = -\frac{1}{2a}(x - a) + a^2$

$y = -\frac{1}{2a}x + \frac{1}{2} + a^2$

when $x = 0$, $y = \frac{1}{2} + a^2 = c$

Hence,

$\frac{1}{2} + a^2 = a^2 + \sqrt{3}a$

$\frac{1}{2} = \sqrt{3}a \Rightarrow a = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$

So $c = \frac{1}{2} + a^2 = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$

$$\boxed{Q = \left(0, \frac{7}{12}\right)}$$