1. (a) (4 pts) Given $f(x) = \left(\sin^2(x) + e^{(5x^4)}\right)^{10}$, find $f'(x)$.

$$\boxed{f'(x) = 10\left(\sin^2(x) + e^{(5x^4)}\right)^9 \left(2\sin(x)\cos(x) + 20x^3 e^{(5x^4)}\right)}$$

(b) (5 pts) Given $y = (x^2+1)^{(\cos(4x))}$, find $\frac{dy}{dx}$.

$$\ln(y) = \cos(4x)\ln(x^2+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x\cos(4x)}{x^2+1} - 4\sin(4x)\ln(x^2+1)$$

$$\boxed{\frac{dy}{dx} = y\left(\frac{2x\cos(4x)}{x^2+1} - 4\sin(4x)\ln(x^2+1)\right) = (x^2+1)^{\cos(4x)}\left(\frac{2x\cos(4x)}{x^2+1} - 4\sin(4x)\ln(x^2+1)\right)}$$

(c) (6 pts) Find the equation of the tangent line to $y^3 + y\sin(x) = \cos(x)$ at the point on the curve where $x = 0$.

$$x = 0 \Rightarrow y^3 + y\sin(0) = \cos(0) \Rightarrow y = 1$$

$$3y^2\frac{dy}{dx} + \frac{dy}{dx}\sin(x) + y\cos(x) = -\sin(x)$$

at $(x,y) = (0,1) \Rightarrow \quad 3\frac{dy}{dx} + 0 + 1\cdot 1 = -0 \Rightarrow \frac{dy}{dx} = -1/3$

$$\boxed{y = -\frac{1}{3}x + 1}$$

2. (7 pts) Use the linear approximation to $f(x) = \tan^{-1}(2x) + \ln(8x^3)$ at $x = \frac{1}{2}$ to estimate the value of $f(0.51)$. (Leave in exact form).

$f(x) = \tan^{-1}(2x) + \ln(8) + 3\ln(x)$ ← still can differentiate without simplifying

$$f'(x) = \frac{2}{1+(2x)^2} + \frac{3}{x}$$

$$f\left(\tfrac{1}{2}\right) = \tan^{-1}(1) + \ln\left(8\cdot\left(\tfrac{1}{2}\right)^3\right) = \pi/4 + 0$$

$$f'\left(\tfrac{1}{2}\right) = \frac{2}{1+1^2} + \frac{3}{(1/2)} = 7$$

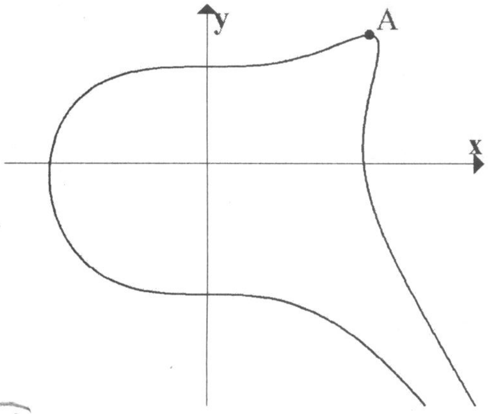
$$y = 7(x - 1/2) + \pi/4$$

$$f(0.51) \approx 7\left(0.51 - \tfrac{1}{2}\right) + \pi/4 = \boxed{0.07 + \pi/4} \approx 0.855398$$

ASIDE:

$$\underbrace{f(0.51) = 0.854706}_{\text{actual}}$$

3. (~~8~~pts) Consider the curve implicitly defined by $\left(x^3 - y^2\right)^2 + e^y = 4$ (shown below). Find the (x, y) coordinates of the point A shown which is the highest point on the curve. (Hint: At this point, there is a horizontal tangent line.).



WANT $\frac{dy}{dx} \overset{?}{=} 0$

$2\left(x^3-y^2\right)\left(3x^2-2y\frac{dy}{dx}\right)+e^y\frac{dy}{dx}=0$

$\frac{dy}{dx}=0 \Leftrightarrow 2\left(x^3-y^2\right)\left(3x^2\right)=0$

$\swarrow$ $\searrow$

$\cancel{x=0}$ OR $x^3-y^2=0$

$\downarrow$

NOT AT A

$x^3-y^2=0 \Rightarrow 0^2+e^y=4 \Rightarrow y=\ln(4)$

and $x=y^{2/3}=\left(\ln(4)\right)^{2/3}$

$$\boxed{A:(x,y)=\left(\left(\ln(4)\right)^{2/3}, \ln(4)\right)}$$

4. (11 pts) **For BOTH parts below**, consider the parametric curve shown which is defined by:

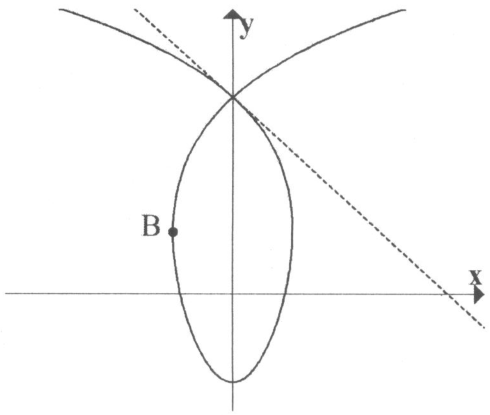
$$x(t) = t^3 - 4t \quad , \quad y(t) = 2\ln(t^2+1) - 1.$$

4 pts (a) The tangent line is vertical at the point B shown in the graph. Find the y-coordinate of the point B. (Leave in exact form)

WANT: $\frac{dx}{dt} = 3t^2 - 4 \overset{?}{=} 0$

$\Rightarrow \quad t = \pm\sqrt{4/3} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2}{3}\sqrt{3}$

$$\boxed{y = 2\ln\left(\frac{4}{3}+1\right) - 1 = 2\ln\left(\frac{7}{3}\right) - 1 = 2\ln(7) - 2\ln(3) - 1}$$



7 pts (b) The curve has one positive y-intercept which it crosses through twice. Find the equation of the tangent line that has negative slope at the positive y-intercept (as shown in the picture)

$y\text{-intercept} \Leftrightarrow x \overset{?}{=} 0 \Leftrightarrow t^3 - 4t \overset{?}{=} 0 \Leftrightarrow t(t^2-4) = 0$

$\cancel{t = 0} \quad$ or $\quad t = -2 \quad$ or $\quad t = 2$

$\cancel{\downarrow}$

$\cancel{y = -1}$

MAKES THIS NEGATIVE

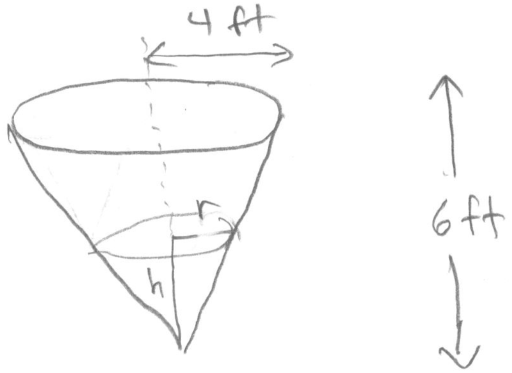
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{4t}{t^2+1}}{3t^2-4} = \frac{4t}{(t^2+1)(3t^2-4)}$

$\left.\frac{dy}{dx}\right|_{t=-2} = \frac{-8}{5 \cdot 8} = -1/5$

$y(-2) = 2\ln(5) - 1$

$$\boxed{y = -\frac{1}{5}(x-0) + 2\ln(5) - 1}$$

5. (8 pts) An inverted cone starts full of water. The height of the cone is 6 ft and the radius is 4 ft. Water leaks out of the bottom at a constant rate of 1 ft^3/min. When the radius is 2 ft, find the rate at which the radius is changing.
(Recall: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$).



GIVEN : $\frac{dV}{dt} = -1 \ \frac{ft^3}{min}$

WANT : $\frac{dr}{dt} = ?$ WHEN $r = 2$

$$\frac{h}{r} = \frac{6}{4} \Rightarrow h = \frac{3}{2} r$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \left(\frac{3}{2} r\right) = \frac{1}{2}\pi r^3$$

$$V = \frac{1}{2}\pi r^3$$

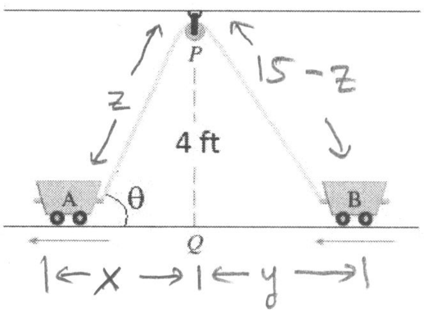
$$\hookrightarrow \frac{dV}{dt} = \frac{3}{2}\pi r^2 \frac{dr}{dt}$$

$$-1 = \frac{3}{2}\pi (2)^2 \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = -\frac{1}{6\pi} \ \frac{ft}{min}} \approx -0.05305 \ ft/min$$

6. (12 pts)

Two carts, A and B, are connected by a rope 15 ft long that passes over a pulley P (see the figure). The point Q is on the floor 4 ft beneath P and between the carts. Cart A is being pulled away from Q at a constant speed of 2 ft/s.



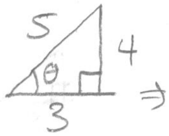
5 pts (a) Let θ be the angle that the rope makes with the ground where it meets cart A (as shown in the picture). Find the rate at which θ is changing at the instant when cart A is 3 ft from Q.

GIVEN: $\frac{dx}{dt} = 2$

WANT: $\frac{d\theta}{dt} = ?$ WHEN $x = 3$

$\tan(\theta) = \frac{4}{x}$

$\frac{d}{dt} \Big[\hookrightarrow \sec^2(\theta)\frac{d\theta}{dt} = -\frac{4}{x^2}\frac{dx}{dt}$ } Plug in ⇒ $\tan(\theta) = \frac{4}{3}$ ⇒



$\Rightarrow \sec\theta = \frac{5}{3}$

$\left(\frac{5}{3}\right)^2 \frac{d\theta}{dt} = -\frac{4}{(3)^2}(2)$

$$\boxed{\frac{d\theta}{dt} = -\frac{8}{25} = -0.32 \ \frac{\text{rad}}{\text{sec}}}$$

7 pts (b) How fast is cart B moving toward Q at the instant when cart A is 3 ft from Q?

GIVEN: $\frac{dx}{dt} = 2$

WANT: $\frac{dy}{dt} = ?$ WHEN $x = 3$

$x^2 + 4^2 = z^2$ and $y^2 + 4^2 = (15 - z)^2$

$\frac{d}{dt}\Big[\hookrightarrow \cancel{2}x\frac{dx}{dt} = \cancel{2}z\frac{dz}{dt}$ and $\cancel{2}y\frac{dy}{dt} = -\cancel{2}(15 - z)\frac{dz}{dt}$

PLUG IN ⇒ $z = 5$ and $y^2 + 16 = 10^2 \Rightarrow y = \sqrt{84} = 2\sqrt{21}$

and $\underbrace{(3)(2) = (5)\frac{dz}{dt}}_{\frac{dz}{dt} = \frac{6}{5}}$ and $\sqrt{84}\frac{dy}{dt} = -(15 - 5)\frac{6}{5} = -12$

THUS, $$\boxed{\frac{dy}{dt} = \frac{-12}{\sqrt{84}} = -\frac{6}{\sqrt{21}} = -\frac{6}{21}\sqrt{21} = -\frac{2}{7}\sqrt{21} \ \frac{\text{ft}}{\text{sec}}}$$

$\approx -1.309 \ \frac{\text{ft}}{\text{sec}}$