1. (12 pts) Compute the derivatives of the following functions. You do not have to simplify your final answer.

(a) $y = \sin\left(\sec\left(\frac{1}{x^2}+1\right)\right)$ CHAIN RULE!

NOTE: $\frac{d}{dx}\left[\frac{1}{x^2}\right] = \frac{d}{dx}\left[x^{-2}\right] = -2x^{-3} = -\frac{2}{x^3}$

$$\boxed{\frac{dy}{dx} = \cos\left(\sec\left(\frac{1}{x^2}+1\right)\right)\sec\left(\frac{1}{x^2}+1\right)\tan\left(\frac{1}{x^2}+1\right)\cdot\frac{-2}{x^3}}$$

(b) $y = x^3 \ln\left(e^{5x} + \sin^4(x)\right)$ PRODUCT RULE, THEN CHAIN RULE!

$$\boxed{\frac{dy}{dx} = 3x^2 \ln\left(e^{5x}+\sin^4(x)\right) + x^3\cdot\frac{5e^{5x}+4\sin^3(x)\cos(x)}{e^{5x}+\sin^4(x)}}$$

(c) $f(x) = \tan^{-1}\left(\sqrt{3x+5^x}\right)$

$$f'(x) = \frac{1}{\sqrt{3x+5^x}^{\,2}+1}\cdot\frac{1}{2\sqrt{3x+5^x}}\cdot\left(3+5^x\ln(5)\right)$$

$$= \frac{3+5^x\ln(5)}{2(3x+5^x+1)\sqrt{3x+5^x}}$$

2. (16 pts) The two parts below are independent of each other.

(a) (8 pts) Find the equation for the tangent line to $y = (3x + 1)^{\sqrt{x}}$ at $x = 1$.

LOGARITHMIC DIFFERENTIATION!

$$\ln(y) = \sqrt{x}\,\ln(3x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln(3x+1) + \sqrt{x}\,\frac{3}{3x+1}$$

$$\frac{dy}{dx} = y\left(\frac{\ln(3x+1)}{2\sqrt{x}} + \frac{3\sqrt{x}}{3x+1}\right)$$

$$y(1) = (3(1)+1)^{\sqrt{1}} = 4$$

$$\frac{dy}{dx}(1) = 4\cdot\left(\frac{\ln(4)}{2} + \frac{3}{4}\right) = 2\ln(4) + 3 \approx 5.772588722$$

$$\boxed{y = (2\ln(4)+3)(x-1)+4}$$

(b) (8 pts) Find all values of t at which the parametric curve $x = 3t^2$, $y = 15t - 3\ln(t)$ has a tangent line with slope 2.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{15 - 3/t}{6t}$$

$$\frac{15 - 3/t}{6t} \overset{?}{=} 2$$

$$15 - 3/t = 12t$$

$$15t - 3 = 12t^2$$

$$0 = 12t^2 - 15t + 3$$

$$0 = 4t^2 - 5t + 1 = (4t-1)(t-1)$$

$$\boxed{t = 1 \quad \text{or} \quad t = 1/4}$$

3. (9 pts) Consider the curve implicitly defined by $4y^3 + x^2\sin(\pi y) - x^2 = 0$. There is one point on the curve that has a y-coordinate of $y = 1$ **and** a negative x-coordinate. Find the equation~~s~~ for the tangent line to the curve at this point.
(Give your numbers simplified in exact form).

$$y = 1 \Rightarrow \quad 4(1)^3 + x^2 \sin(\pi) - x^2 = 0$$

$$\Rightarrow \quad 4 - x^2 = 0 \Rightarrow \quad (2-x)(2+x) = 0$$

$$x = \pm 2$$

x negative $\Rightarrow x = -2$ POINT: $(-2, 1)$

$$12y^2 \frac{dy}{dx} + 2x\sin(\pi y) + \pi x^2 \cos(\pi y)\frac{dy}{dx} - 2x = 0$$

$$\left(12y^2 + \pi x^2\cos(\pi y)\right)\frac{dy}{dx} = 2x - 2x\sin(\pi y)$$

$$\frac{dy}{dx} = \frac{2x(1-\sin(\pi y))}{12y^2 + \pi x^2 \cos(\pi y)}$$

SAME

at $(-2,1)$: $\dfrac{dy}{dx} = \dfrac{2(-2)(1-\sin(\pi))}{12(1)^2 + \pi(-2)^2\cos(\pi)} = \dfrac{-4}{12 - 4\pi} = \dfrac{4}{4\pi - 12}$

≈ 7.062513306

$$\boxed{y = \frac{4}{4\pi - 12}(x+2) + 1}$$

4. (12 pts) Larry Bernandez throws a baseball whose location (viewed from the side) is given by the equations:

$$x(t) = 80t\ ,\ \ y(t) = -16t^2 + 8t + 6,$$

where t is in seconds since it was thrown and distances are in feet.

(a) (5 pts) Find the **coordinates** of the ball at the instance when its vertical velocity is zero.

$y'(t) = -32t + 8 \overset{?}{=} 0 \Rightarrow$ $32t = 8$

$t = 8/32 = 1/4$ sec

$x(1/4) = 80(1/4) = 20$

$y(1/4) = -16\left(\frac{1}{4}\right)^2 + 8(1/4) + 6 = -1 + 2 + 6 = 7$

$\boxed{(20, 7)}$

(b) (7 pts) Recall, the speed is given by $\sqrt{(x'(t))^2 + (y'(t))^2}$ ft/sec. Find the speed of the ball **and** the equation of the tangent line at the instant when the ball reaches the point $(x, y) = (60, 3)$.

$80t = 60 \Rightarrow t = \frac{60}{80} = 3/4$ So $t = 3/4$ sec

$-16t^2 + 8t + 6 = 3$ ← ALSO WORKS HERE

SPEED $= \sqrt{(80)^2 + (-32t + 8)^2}$

$= \sqrt{(80)^2 + (-24 + 8)^2}$ at $t = 3/4$

$= \sqrt{80^2 + (-16)^2} = \sqrt{6656}$ $\left(= \sqrt{16^2(5^2 + 1^2)} = 16\sqrt{26}\right)$

≈ 81.5843122

$\frac{dy}{dx} = \frac{-32t + 8}{80}$

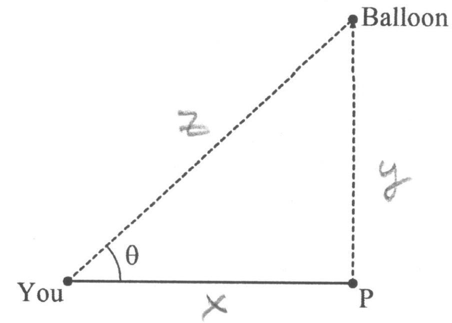
$\frac{dy}{dx} = \frac{-24 + 8}{80} = \frac{-16}{80} = -\frac{1}{5}$ at $t = 3/4$

SPEED $\boxed{\sqrt{6656}}$ ft/sec

TANGENT LINE EQUATION: $\boxed{y = -\frac{1}{5}(x - 60) + 3}$

5. (11 pts)

At time $t = 0$ seconds, a balloon is released from the ground from a point P which is 332 feet away from you. You are walking toward the point P at the constant rate of 4 feet per second and the balloon is rises vertically at the constant rate of 50 feet per second.

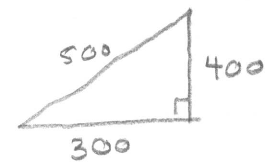


(a) (6 pts) At what rate is the straight line distance between you and the balloon changing when the balloon is exactly 400 feet high? (Hint: Note it takes $t = 8$ seconds for the balloon to get to 400 feet high.)

NOTE: $t = 8 \Rightarrow$ YOU'VE TRAVELED $4 \frac{ft}{sec} \cdot 8\,sec = 32$ ft ; $332 - 32 = 300$

SO AT THAT INSTANT THE PICTURE LOOKS LIKE THIS

$\sqrt{300^2 + 400^2} = 500$



WE ALSO KNOW $\frac{dx}{dt} = -4$, $\frac{dy}{dt} = 50$, $\frac{dz}{dt} = ?$

IN GENERAL, $x^2 + y^2 = z^2$

THUS, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$

SO $300(-4) + 400(50) = 500\frac{dz}{dt}$

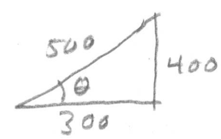
$18800 = 500\frac{dz}{dt}$

$$\boxed{\frac{dz}{dt} = \frac{18800}{500} = \frac{188}{5} = 37.6 \text{ ft/sec}}$$

(b) (5 pts) Let θ be the angle of inclination between your line of sight to the balloon and the ground (see picture). At what rate is θ changing when the balloon is exactly 400 feet high? (Give in units of rad/sec).

YOU CAN USE ANY OF THE TRIG FUNCTIONS HERE. WITH $\tan(\theta)$ THE SOL'N LOOKS LIKE:

$\tan(\theta) = \frac{y}{x} \Rightarrow \sec^2(\theta)\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$



FROM HERE $\sec(\theta) = \frac{HYP}{ADJ} = \frac{500}{300} = 5/3$

THUS, $\left(\frac{5}{3}\right)^2 \frac{d\theta}{dt} = \frac{(300)(50) - (400)(-4)}{(300)^2} = \frac{166\cancel{00}}{900\cancel{00}}$

$$\boxed{\frac{d\theta}{dt} = \frac{166}{900} \cdot \frac{9}{25} = \frac{166}{2500} = \frac{83}{1250} = 0.0664 \ \frac{rad}{sec}}$$