1. (12 pts) Determine the values of the following limits or state that the limit does not exist. If it is correct to say that the limit equals ∞ or $-\infty$, then you should do so. **In all cases, show your work/reasoning. You must use algebraic methods where available. And explain in words your reasoning if an algebraic method is not available.**

(a) $\lim_{x\to 2} \dfrac{\frac{1}{x^2+1} - \frac{1}{5}}{x-2} \cdot \dfrac{5(x^2+1)}{5(x^2+1)} = \lim_{x\to 2} \dfrac{5-(x^2+1)}{5(x-2)(x^2+1)}$

$= \lim_{x\to 2} \dfrac{4-x^2}{5(x-2)(x^2+1)}$

$= \lim_{x\to 2} \dfrac{(2-x)(2+x)}{5(x-2)(x^2+1)}$

$= \lim_{x\to 2} \dfrac{-(2+x)}{5(x^2+1)} = \boxed{\dfrac{-4}{25}}$

NOTE: $\dfrac{2-x}{x-2} = \dfrac{-(x-2)}{(x-2)} = -1$

(b) $\lim_{t\to 7} \dfrac{|10-t| + \cos(\pi t) - 30}{(t-7)^2}$

THE NUMERATOR IS APPROACHING $|10-7| + \cos(7\pi) - 30 = -28$, AND THE DENOMINATOR IS APPROACHING 0 THROUGH POSITIVE VALUES.

THUS, THE LIMIT IS $\boxed{-\infty}$

(c) $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{3x-2} - \underbrace{\dfrac{4x}{6x-4}}_{\frac{2x}{3x-2}} = \lim_{x\to\infty} \dfrac{\sqrt{x^2+1} - 2x}{3x-2} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim_{x\to\infty} \dfrac{\frac{\sqrt{x^2+1}}{x} - 2}{3 - \frac{2}{x}}$

For $x>0$, $x = \sqrt{x^2}$

$= \lim_{x\to\infty} \dfrac{\sqrt{1+\frac{1}{x^2}} - 2}{3 - \frac{2}{x}}$

$= \dfrac{\sqrt{1+0} - 2}{3-0} = \boxed{\dfrac{-1}{3}}$

2. (12 pts) Find the indicated derivatives of the following functions. (You do not have to simplify your final answer).

(a) $y = \frac{4}{x^2} + \frac{x^3}{6} + \frac{3}{e^x} + \frac{e^x}{2}$, find y'

INSTEAD ON $\frac{3}{e^x}$ YOU COULD USE THE QUOTIENT RULE

$$\frac{e^x(0) - 3e^x}{e^{2x}} = -3e^{-x}$$

$$y = 4x^{-2} + \frac{1}{6}x^3 + 3e^{-x} + \frac{1}{2}e^x$$

CHAIN RULE

$$\boxed{\begin{aligned} y' &= -8x^{-3} + \frac{3}{6}x^2 - 3e^{-x} + \frac{1}{2}e^x \\ &= -\frac{8}{x^3} + \frac{1}{2}x^2 - 3e^{-x} + \frac{1}{2}e^x \end{aligned}}$$

(b) $y = \sqrt{9x^3}\sec(x) - 16$, find $\frac{dy}{dx}$.

NOTE: $\sqrt{9x^3} = 3x^{3/2}$

$$y = \underbrace{3x^{3/2}}_{F}\underbrace{\sec(x)}_{S} - 16$$

PRODUCT RULE

$$\boxed{\begin{aligned} \frac{dy}{dx} &= 3x^{3/2}\sec(x)\tan(x) + \frac{9}{2}x^{1/2}\sec(x) - 0 \\ &= 3\sqrt{x}\sec(x)\left(x\tan(x) + \frac{3}{2}\right) \end{aligned}}$$

(c) $f(x) = \left(3 - 5\sqrt[3]{x}\right)^2$, find $f'(x)$.

EXPANDING:

$$f(x) = 9 - 30x^{1/3} + 25x^{2/3}$$

$$f'(x) = 0 - 10x^{-2/3} + \frac{50}{3}x^{-1/3}$$

OR

CHAIN RULE:

$$f(x) = (3 - 5x^{1/3})^2$$

$$f'(x) = 2(3 - 5x^{1/3})\left(-\frac{5}{3}x^{-2/3}\right)$$

3. Assume the height, in feet, of a particular particle is given by $f(t) = \sqrt{2t+1}$ where t is in seconds.

(a) (5 pts) Find and *completely simplify* the expression $\dfrac{f(t+h) - f(t)}{h}$.

$$\frac{\left(\sqrt{2(t+h)+1} - \sqrt{2t+1}\right)}{h} \cdot \frac{\sqrt{2(t+h)+1} + \sqrt{2t+1}}{\sqrt{2(t+h)+1} + \sqrt{2t+1}} \quad \Big\} \text{ CONJUGATE}$$

$$= \frac{(2(t+h)+1) - (2t+1)}{h\left(\sqrt{2t+2h+1} + \sqrt{2t+1}\right)} = \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2t+2h+1} + \sqrt{2t+1}\right)}$$

$$= \boxed{\frac{2}{\sqrt{2t+2h+1} + \sqrt{2t+1}}}$$

(b) (3 pts) Find the average speed of the particle over the interval from $t = 1$ to $t = 4$ seconds. (Include units in your answer)

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\sqrt{2(4)+1} - \sqrt{2(1)+1}}{3} = \boxed{\frac{3 - \sqrt{3}}{3} \ \frac{\text{ft}}{\text{sec}}}$$

WHICH IS THE SAME AS TAKING $t = 1$ AND $h = 3$ IN PART (a)

$$= \frac{2}{\sqrt{2(1)+2(3)+1} + \sqrt{2(1)+1}} = \boxed{\frac{2}{3+\sqrt{3}} \ \frac{\text{ft}}{\text{sec}}}$$

SAME

NOTE:
$$\frac{2}{3+\sqrt{3}} \cdot \frac{(3-\sqrt{3})}{(3-\sqrt{3})} = \frac{6 - 2\sqrt{3}}{9-3} = \frac{6-2\sqrt{3}}{6} = \frac{3-\sqrt{3}}{3}$$

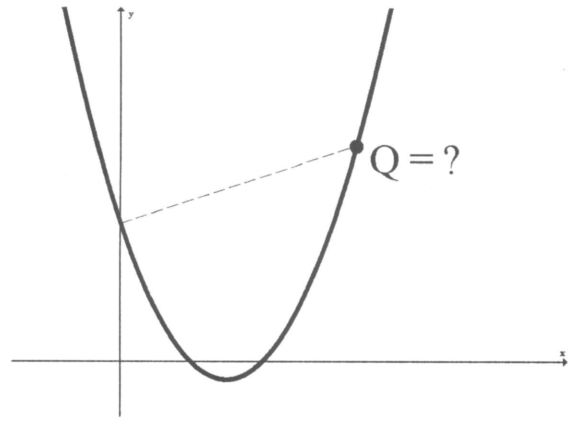
(c) (2 pts) Find the instantaneous speed of the particle at $t = 1$ seconds. (Include units in your answer)

LET $h \to 0$ IN PART (a) TO GET $f'(t) = \dfrac{2}{\sqrt{2t+1} + \sqrt{2t+1}} = \dfrac{2}{2\sqrt{2t+1}}$

$$f'(t) = \frac{1}{\sqrt{2t+1}}$$

THUS, $f'(1) = \dfrac{1}{\sqrt{2(1)+1}} = \boxed{\dfrac{1}{\sqrt{3}} \ \text{ft}/\text{sec}}$

4. (a) (7 pts) Viewed from above, you are walking from left to right along the curve $y = x^2 - 3x + 2$ in the xy-plane. When you get to the point $(0, 2)$, you leave the path and follow the normal line for a shortcut. Find the x and y coordinates of the point, Q, where you meet up with the path again.



$y' = 2x - 3$

"TANGENT SLOPE AT $(0,2)$" $= 2(0) - 3 = -3$

"NORMAL SLOPE AT $(0,2)$" $= -\frac{1}{-3} = \frac{1}{3}$

NORMAL LINE AT $(0,2)$:

$y = \frac{1}{3}x + 2$

INTERSECTION: $\frac{1}{3}x + 2 = x^2 - 3x + 2$

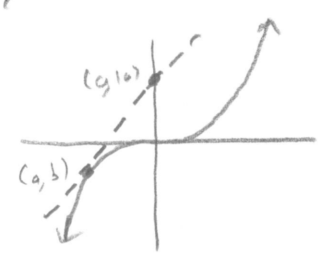
$0 = x^2 - 3x - \frac{1}{3}x$

$0 = x^2 - \frac{10}{3}x = x\left(x - \frac{10}{3}\right)$

$x = 0$ ✓ or $\boxed{x = 10/3}$

$\boxed{y = \frac{1}{3}\left(\frac{10}{3}\right) + 2 = \frac{10}{9} + \frac{18}{9} = \frac{28}{9}}$

(b) (7 pts) Find the x and y coordinates of a point P on the curve $y =$ [redacted] x^3 at which the tangent line at the point P has a y-intercept of 10.



① (a,b) IS ON THE CURVE $\Rightarrow$ $b = a^3$

② "SLOPE FROM (a,b) TO $(0,10)$" $=$ "SLOPE OF TANGENT TO $y = x^3$ AT (a,b)"

$y' = 3x^2$

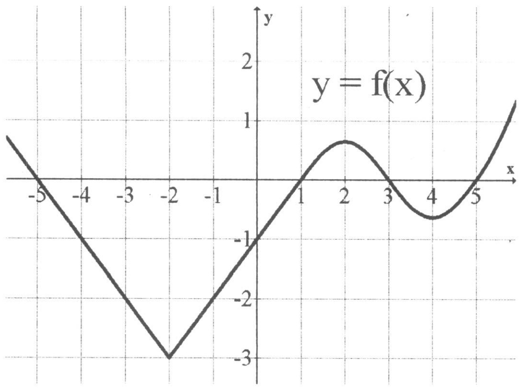
$\Rightarrow \frac{b-10}{a-0} = 3a^2$

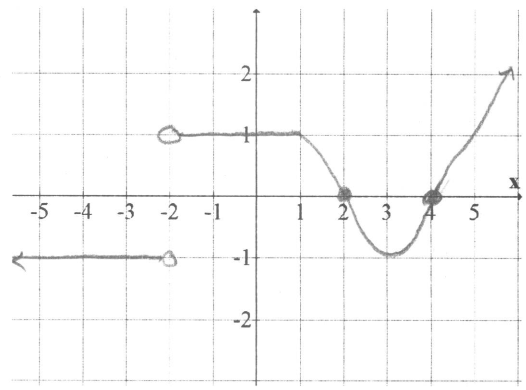
$b - 10 = 3a^3$

① & ② $\Rightarrow$ $a^3 - 10 = 3a^3 \Rightarrow -10 = 2a^3 \Rightarrow -5 = a^3$

$\boxed{\begin{array}{l} a = (-5)^{1/3} = -\sqrt[3]{5} \\ b = ((-5)^{1/3})^3 = -5 \end{array}}$

5. (a) (4 pts) The graph of a function $y = f(x)$ is shown. Sketch a rough graph of the derivative $y' = f'(x)$.





(b) For a constant c, consider the function $g(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & \text{if } x < 3; \\ cx^2 + 10, & \text{if } x \geq 3. \end{cases}$

i. (5 pts) Find the value c that will make this function continuous at $x = 3$.

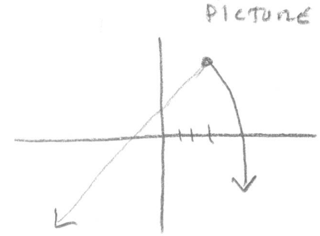
NOTE: FOR $x < 3$, $\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{(x-3)} = x+3$

$\lim_{x \to 3^-} g(x) = 3 + 3 = 6$

WANT EQUAL

$\lim_{x \to 3^+} g(x) = c(3)^2 + 10 = 9c + 10$

NOTE: HERES THE PICTURE



$9c + 10 = 6$

$9c = -4$

$\boxed{c = -4/9}$

ii. (3 pts) For the value of c you found in the previous part is the function $f(x)$ differentiable at $x = 3$? (Explain)

$\frac{x^2-9}{x-3} = x+3$

THE DERIVATIVE FROM THE LEFT

$\lim_{h \to 0^-} \frac{g(3+h) - g(3)}{h}$ WILL EQUAL $\frac{d}{dx}(x+3) = 1$ at $x = 3$.

THE DERIVATIVE FROM THE RIGHT

$\lim_{h \to 0^+} \frac{g(3+h) - g(3)}{h}$ WILL EQUAL $\frac{d}{dx}\left(-\frac{4}{9}x^2 + 10\right) = -\frac{8}{9}x$ WHICH IS $-\frac{8}{3}$ at $x = 3$

$\boxed{\text{THE SLOPES ARE NOT EQUAL SO } f(x) \text{ IS NOT DIFFERENTIABLE AT } x = 3. \text{ (SHARP CORNER)}}$