

Week 5 Overview

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections.

3.5: Implicit Differentiation

1. First understand the idea of implicitly defined functions: An equation involving x and y determines some set of points (some *curve*) in the xy -plane. The curve may not define a function (there may be several y values for each x value). However, the curve can be ‘cut’ into pieces that do define functions in the form $y = f(x)$. We say that $y = f(x)$ is one of the functions that is *implicitly* defined by the equation. It is important to realize that in many situations it is difficult (if not impossible) to solve for y in terms of x in some compact way.
2. The good news is that even if we can’t solve for y in terms of x , we still can get information about the slope of the tangent lines (*i.e.* we still can get the derivative). Here is how we do it:
 - (a) First, remember to think of y as one of the implicitly defined functions of x . So if it helps, rewrite y as $y(x)$.
 - (b) Differentiate both sides of the equation in terms of x . Remember that $y = y(x)$, so you must differentiate terms involving y as well. And you must remember to differentiate y (*i.e.* use the chain rule) where appropriate and write y' or $\frac{dy}{dx}$.
 - (c) Solve for $\frac{dy}{dx}$ in terms of x and y .
3. Know how to use implicit differentiation to find derivatives of inverse functions. In general, if $y = f^{-1}(x)$ is the inverse of some function with a known derivative, then we can rewrite the relationship as $f(y) = x$ and differentiate implicitly. This gives: $f'(y)\frac{dy}{dx} = 1$, so

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}. \text{ Thus,}$$

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}.$$

It is no so important to memorize this formula, but you should understand the technique.

4. As a quick application that we didn’t do in class, consider $y = x^{1/3}$, so $y^3 = x$ which means $3y^2y' = 1$, so $y' = 1/(3y^2) = 1/(3x^{2/3})$. We certainly wouldn’t do this since we know the power rule, but it shows how this can be used.
5. Inverse Trig: The big application in 3.5 of this inverse idea was inverse trig functions. With trig functions we are able to exploit the relationships between the trig functions (using a triangle) to simplify our answer into a non-trig form involving x . This is a useful idea, so make sure you know how to use it. Here are the derivatives:

$\frac{d}{dx} [\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx} [\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx} [\tan^{-1}(x)] = \frac{1}{1+x^2}$	$\frac{d}{dx} [\cot^{-1}(x)] = \frac{-1}{1+x^2}$
$\frac{d}{dx} [\sec^{-1}(x)] = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx} [\csc^{-1}(x)] = \frac{-1}{x\sqrt{x^2-1}}$

You should certainly know the domains and ranges of these functions. This is important because you have to remember for instance that $\sin^{-1}(1/2)$ only gives ONE of the infinitely many solutions to $\sin(x) = 1/2$. You should know which answer it gives and how to find the other solutions.

6. Let me expand on this last point with a quick digression into solving trigonometric equations. Say we want to solve $\cos(BLAH) = 1/2$. If you think about the graph of $y = \cos(x)$ there are TWO solutions every period (we call these the principal and symmetric solutions). A calculator will give you a principal solution (which for $\cos(x)$ is between 0 and π). You have to exploit the symmetries in the $\cos(x)$ graph to get the other solutions. So you would proceed as follows to solve $\cos(BLAH) = 1/2$:

- Principal solution: $\text{BLAH} = \frac{\pi}{3}$ radians
- A symmetric solution: The first high point on the $\cos(x)$ wave is at $x = 0$, so there should be another solution equidistance on the other side of this high point. That solution would be: $\text{BLAH} = -\frac{\pi}{3}$.
- Other solutions: Now these same two types of solutions will repeat every period so we can add/subtract 2π any whole number of times to get other solutions.
- Thus, we can summarize that all solutions are of the form:
 $\text{BLAH} = \frac{\pi}{3} + 2n\pi$, or $\text{BLAH} = -\frac{\pi}{3} + 2n\pi$ for some integer n (*i.e.* positive or negative whole number).

3.6: Derivatives of Logarithms and Implicit Differentiation

1. Before we can do anything serious with exponentiation and logarithms, you need to know how to manipulate exponents and logarithms. Here are some facts you should know:

- $\ln(x) = y$ is the same relationship as $x = e^y$ (this is the definition).
- Thus, $\ln(e^y) = y$ and $e^{\ln(x)} = x$ (as they are inverses).
- $\ln(1) = 0$ which says the same thing as $1 = e^0$.
- $\ln(e) = 1$ which says the same thing as $e = e^1$.
- $\ln(a^b) = b \ln(a)$ which comes from the exponent fact $(e^x)^y = e^{xy}$.
- $\ln(ab) = \ln(a) + \ln(b)$ which comes from the exponent fact $e^x e^y = e^{x+y}$
- $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$ which comes from the exponent fact $\frac{e^x}{e^y} = e^{x-y}$
- You should know how to work with fractional exponents and negative exponents.
- Similarly, $\log_a(x) = y$ is the same relationship as $x = a^y$. And similar rules hold.

2. Using the implicit differentiation ideas of the 3.5 we are able to find the derivatives of $y = \ln(x)$ and $y = \log_a(x)$. The derivatives we found are:

$$\frac{d}{dx}[\log_a(x)] = \frac{1}{x \ln(a)} \quad \text{and} \quad \frac{d}{dx}[\ln(x)] = \frac{1}{x}.$$

3. As always we can use the chain rule in conjunction with this new rule, which gives the fact:

$$\frac{d}{dx}[\ln(g(x))] = \frac{g'(x)}{g(x)}.$$

4. Logarithmic Differentiation: By using the logarithm rules, we see that the logarithm of an expression essential turns multiplication/division problems into addition/subtraction problems and exponent problems into multiplication problems. We can exploit this fact as follows:

If we encounter a function with the variable x in the base AND in the exponent (or if we encounter a particularly messy quotient or product), we can use the following method:

- Take the natural logarithm of both side. Simplify with logarithm rules.
- Implicitly differentiate both sides.
- Solve for $\frac{dy}{dx}$
- Write in terms of x (if possible).

With this method and the others we have learned, we can now differential nearly any function we could encounter.