1. (13 pts) (You don't need to simplify your derivatives)

(a) Let $y = \tan^5(e^{3x})$. Find $\frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = 5\tan^4(e^{3x})\sec^2(e^{3x})\,e^{3x}\cdot 3 = 15e^{3x}\tan^4(e^{3x})\sec^2(e^{3x})}$$

(b) Let $g(x) = \frac{x}{2} + \arctan(2x)$.
Find the value(s) of x at which the slope of the tangent line to $g(x)$ is 1.

$$g'(x) = \frac{1}{2} + \frac{2}{1+(2x)^2} = \frac{1}{2} + \frac{2}{1+4x^2} \stackrel{?}{=} 1$$

$$\Rightarrow \frac{2}{1+4x^2} \stackrel{?}{=} 1/2$$

$$\Rightarrow 4 = 1 + 4x^2$$

$$\Rightarrow 3 = 4x^2$$

$$x^2 = 3/4$$

$$\boxed{x = \pm\frac{\sqrt{3}}{2}}$$

(c) Find the value(s) of x at which $f(x) = \sqrt{4x - x^{4/3}}$ has a horizontal tangent line.

$$f'(x) = \frac{4 - \frac{4}{3}x^{1/3}}{2\sqrt{4x - x^{4/3}}} \stackrel{?}{=} 0$$

$$\Rightarrow 4 - \frac{4}{3}x^{1/3} \stackrel{?}{=} 0$$

$$\Rightarrow 4 = \frac{4}{3}x^{1/3}$$

$$\Rightarrow 3 = x^{1/3}$$

$$\Rightarrow \boxed{x = 27}$$

2. (12 pts) For all parts below, consider a particle moving in the xy-plane such that its location at time t seconds is given by:

$$x(t) = \frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t + 2 \ , \quad y(t) = \ln(3t+1) + 4^t - \ln(4)t + 5,$$

where x and y are in feet.

(a) Find the following:

i. The formula for the vertical velocity in terms of time t.

$$\boxed{\frac{dy}{dt} = \frac{3}{3t+1} + 4^t \ln(4) - \ln(4)}$$

ii. The speed of the particle at time $t = 0$. (include units)

$$\sqrt{(x'(0))^2 + (y'(0))^2} = \sqrt{(10)^2 + (3 + \ln(4) - \ln(4))^2}$$
$$= \boxed{\sqrt{109} \ \text{ft/sec}}$$

iii. The equation for the tangent line to the curve at time $t = 0$ (in the form $y = mx + b$).

$$x'(t) = t^2 - 7t + 10$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}\Big|_{t=0} = \frac{3}{10}$$

$$x(0) = 2$$

$$y(0) = 4^0 + 5 = 6$$

$$\boxed{\begin{aligned} y &= \frac{3}{10}(x-2) + 6 \\ y &= \frac{3}{10}x - \frac{3}{5} + 6 \\ y &= \frac{3}{10}x + \frac{27}{5} \end{aligned}}$$

(b) Find all times, t, then the curve has a **vertical** tangent line.

$$t^2 - 7t + 10 \stackrel{?}{=} 0$$

$$(t-2)(t-5) \stackrel{?}{=} 0$$

$$\boxed{t = 2 \quad \text{or} \quad t = 5}$$

3. (14 pts)

(a) Find the equation of the tangent line to $y = (2x+1)^{\cos(\pi x)}$ at $x = 1$.

$$y(1) = (2(1)+1)^{\cos(\pi)} = 3^{-1} = 1/3$$

$$\ln(y) = \cos(\pi x)\ln(2x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = -\pi\sin(\pi x)\ln(2x+1) + \cos(\pi x)\frac{2}{2x+1}$$

$$\frac{dy}{dx} = y\left(-\pi\sin(\pi x)\ln(2x+1) + \frac{2\cos(\pi x)}{2x+1}\right)$$

at $x=1, y=1/3$: $\frac{dy}{dx} = \frac{1}{3}\left(-\pi(0)\ln(3) + \frac{2(-1)}{3}\right) = -\frac{2}{9}$

$$\boxed{y = -\frac{2}{9}(x-1) + 1/3 = -\frac{2}{9}x + 5/9}$$

(b) The implicit defined curve $(x^2+y)^2+xy^5 = 4$ has only one point where it crosses the positive y-axis. Find the equation of the tangent line at the positive y-intercept.

$x=0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2 \Rightarrow y = 2$

$$2(x^2+y)\left(2x+\frac{dy}{dx}\right) + y^5 + 5xy^4\frac{dy}{dx} = 0$$

$x=0, y=2 \Rightarrow$ $2(0+2)\left(0+\frac{dy}{dx}\right) + 32 + 0 = 0$

$$4\frac{dy}{dx} + 32 = 0$$

$$\frac{dy}{dx} = -8$$

$$\boxed{y = -8x + 2}$$

4. (8 pts) A spherical snowball is melting. At the moment when the radius is 5 cm, its surface area is decreasing at a rate of 3 cm^2/min. Find the rate at which the **volume** is changing at this same moment. (Include units in your final answer and indicate if your answer is positive or negative).

Recall: Volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the surface area of a sphere is $S = 4\pi r^2$.

KNOW: $\frac{dS}{dt} = -3$ when $r = 5$

WANT: $\frac{dV}{dt} = ?$ when $r = 5$.

① $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

at this instant $\rightarrow -3 = 8\pi(5)\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = -\frac{3}{40\pi}$ cm/min

② $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

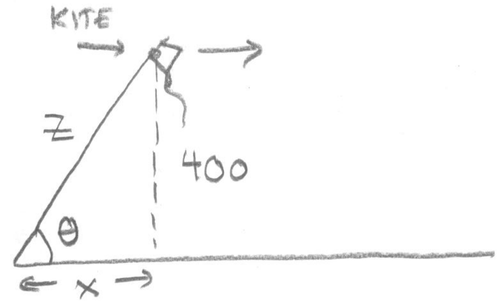
at this instant $\rightarrow \frac{dV}{dt} = 4\pi(5)^2 \cdot \left(-\frac{3}{40\pi}\right) = -\frac{15}{2}$

$$\boxed{\frac{dV}{dt} = -\frac{15}{2} = -7.5 \ \frac{\text{cm}^3}{\text{min}}}$$

5. (12 pts) A kite is in the air at an altitude of 400 feet and is being blown *horizontally* at the constant rate of 10 feet per second away from the person holding the kite string at ground level. (Thus, the kite is remaining at a constant altitude of 400 feet).
For both parts: Include units for your final answers and indicate if your answers are positive or negative. Your final answers should be simplified numbers/fractions.

(a) At what rate is the string being let out when 500 feet of string is already out?

KNOW: $\frac{dx}{dt} = 10$



WANT: $\frac{dz}{dt} = ?$ WHEN $z = 500$

① $x^2 + 400^2 = z^2 \Rightarrow 2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

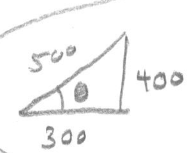
WHEN $z = 500$, ① $\Rightarrow$ $x^2 + 400^2 = 500^2 \Rightarrow x = 300$

and $300 \cdot 10 \stackrel{?}{=} 500 \cdot \frac{dz}{dt}$

$$\boxed{\frac{dz}{dx} = \frac{300 \cdot 10}{500} = 6 \ \frac{ft}{sec}}$$

(b) Let θ be the angle the string makes with the ground at a given time. At what rate is θ changing at the instant when 500 feet of string is already out?

KNOW: $\frac{dx}{dt} = 10$



$\Rightarrow \sec\theta = \frac{500}{300} = 5/3$

WANT: $\frac{d\theta}{dt} = ?$ WHEN $z = 500$

① $\tan\theta = \frac{400}{x}$ $\leftarrow$ (OR $\sin\theta = \frac{400}{z}$ OR $\cos\theta = \frac{x}{z}$)

$\Rightarrow \sec^2\theta \frac{d\theta}{dt} = -\frac{400}{x^2}\frac{dx}{dt}$ $\left(\frac{400}{x} = 400x^{-1} \ \frac{d}{dx} \ -400x^{-2}\right)$

$\left(\frac{5}{3}\right)^2 \frac{d\theta}{dt} = -\frac{400}{(300)^2} \cdot 10$

$$\boxed{\frac{d\theta}{dt} = -\frac{400 \cdot 10 \cdot 3^2}{\cancel{300} \cdot \cancel{300} \cdot 5^2} = -\frac{4}{250} = \frac{-2}{125} \ \frac{rad}{sec}}$$

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