1. (8 pts) Determine the values of the following limits or state that the limit does not exist. If it is correct to say that the limit equals ∞ or $-\infty$, then you should do so. **In all cases, show your work/reasoning. You must use algebraic methods where available. And explain in words your reasoning if an algebraic method is not available.**

(a) $\lim_{x\to 3^-} \frac{x^2-4}{x-3} = \boxed{-\infty}$

Numerator: $\lim_{x\to 3^-} x^2 - 4 = 5$

Denominator: $\lim_{x\to 3^-} x - 3 = 0$

↳ gets closer and closer to zero through negative numbers.

(b) $\lim_{t\to 0} \frac{2}{t(1+3t)^2} - \frac{2}{t} = \lim_{t\to 0} \frac{2-2(1+3t)^2}{t(1+3t)^2}$

$= \lim_{t\to 0} \frac{2-2(1+6t+9t^2)}{t(1+3t)^2}$

$= \lim_{t\to 0} \frac{\cancel{2}-\cancel{2}-12t-18t^2}{t(1+3t)^2}$

$= \lim_{t\to 0} \frac{-12-18t}{(1+3t)^2} = \frac{-12}{1} = \boxed{-12}$

(c) $\lim_{x\to\infty} \frac{\left(4x^2 - \sqrt{x^4+2x-1}\right)}{\left(e^{-x}+3x-\sqrt{5x^3+16x^4}\right)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{4-\sqrt{1+\frac{2}{x^3}-\frac{1}{x^4}}}{\frac{1}{x^2e^x}+\frac{3}{x}-\sqrt{\frac{5}{x}+16}}$

NOTE: $x^2 = \sqrt{x^4}$

$= \frac{4-\sqrt{1+0-0}}{0+0-\sqrt{0+16}}$

$= \boxed{\frac{3}{-4}}$

2. (9 pts)

(a) Compute $\lim\limits_{t\to\pi/2} \dfrac{\left(\sin(t) - \sqrt{\sin^2(t) + 4\cos^2(t)}\right)}{3\cos^2(t)}$ $\dfrac{\left(\sin(t) + \sqrt{\sin^2(t)+4\cos^2(t)}\right)}{\left(\sin(t) + \sqrt{\sin^2(t)+4\cos^2(t)}\right)}$

$$= \lim_{t\to\frac{\pi}{2}} \frac{\cancel{\sin^2(t)} - \cancel{\sin^2(t)} - 4\cancel{\cos^2(t)}}{3\cancel{\cos^2(t)}\left(\sin(t) + \sqrt{\sin^2(t)+4\cos^2(t)}\right)}$$

$$= \frac{-4}{3\left(1+\sqrt{1+0}\right)} = \boxed{\frac{-4}{6} = -\frac{2}{3}}$$

(b) Let $y = \dfrac{5}{2x} + \dfrac{4x}{5\sqrt[4]{x}} - \dfrac{4\tan(x)}{x^5}$. Find $\dfrac{dy}{dx}$. (You don't have to simplify)

$$y = \frac{5}{2}x^{-1} + \frac{4}{5}x^{3/4} - \frac{4\tan(x)}{x^5}$$

$$\boxed{\begin{aligned}\frac{dy}{dx} &= -\frac{5}{2}x^{-2} + \frac{3}{5}x^{-1/4} - \frac{\left(4x^5\sec^2(x) - 20x^4\tan(x)\right)}{x^{10}} \\ &= -\frac{5}{2x^2} + \frac{3}{5\sqrt[4]{x}} - \frac{\left(4x\sec^2(x) - 20\tan(x)\right)}{x^6}\end{aligned}}$$

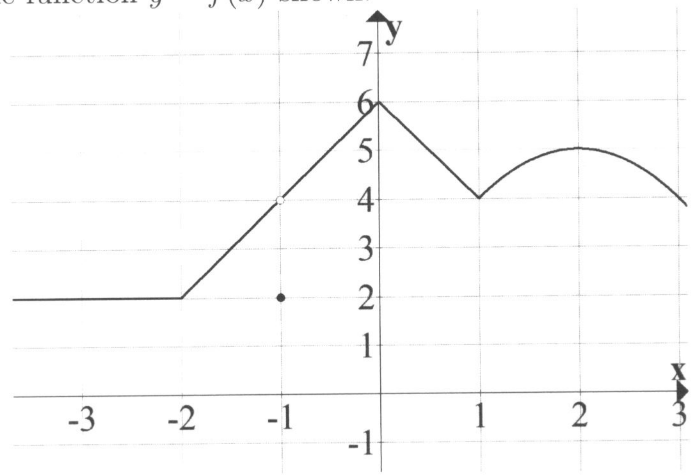
(c) Let $f(t) = \underbrace{5te^t}_{F}\underbrace{\cos(t)}_{S}$. Find the slope of the tangent line to $f(t)$ at $t = \pi$.

$$f'(t) = -5te^t\sin(t) + \left(5e^t + 5te^t\right)\cos(t)$$

$$f'(\pi) = -5\pi e^\pi\sin(\pi) + \left(5e^\pi + 5\pi e^\pi\right)\cos(\pi)$$

$$\boxed{\begin{aligned}&= 0 - \left(5e^\pi + 5\pi e^\pi\right) \\ &= -5e^\pi(1+\pi)\end{aligned}}$$

3. (10 pts) Consider the function $y = f(x)$ shown:



Use the graph to estimate/compute the answer to the following:

(a) Find the solution(s) to $f'(x) = 0$.

$\boxed{x = 2 \quad \text{and} \quad x < -2}$

(b) Name the value(s) of x at which $y = f(x)$ is not differentiable.

$\boxed{x = -2, -1, 0, 1}$

(c) Compute $\lim_{x \to -1} \left(f(x) + \frac{|x-5|}{\sec(\pi x/6)} + \frac{\sin(x+1)}{x+1} \right)$.

$\sec(-\pi/6) = \frac{1}{\cos(-\pi/6)} = \frac{1}{\sqrt{3}/2}$

$$4 + \frac{6}{\sec(-\pi/6)} + \lim_{x \to -1} \frac{\sin(x+1)}{x+1}$$

$$4 + \frac{6}{(2/\sqrt{3})} + 1 = \boxed{5 + 3\sqrt{3}}$$

(d) If $g(x) = \frac{f(x)}{x^2}$, then find value of $y = g'(x)$ at $x = \frac{1}{2}$.

$$g'(x) = \frac{x^2 f'(x) - 2x f(x)}{x^4} = \frac{x f'(x) - 2 f(x)}{x^3}$$

$$\begin{cases} f'(\tfrac{1}{2}) = \frac{4-6}{1-0} = -2 \\ f(\tfrac{1}{2}) = 5 \end{cases}$$

$$g'(\tfrac{1}{2}) = \frac{\frac{1}{2} f'(\frac{1}{2}) - 2 f(\frac{1}{2})}{1/8}$$

$$= \frac{\frac{1}{2}(-2) - 2(5)}{1/8} = \frac{-11}{(1/8)} = \boxed{-88}$$

4. (10 pts) Consider $f(x) = \begin{cases} bx^2 + 3ax - 10 & \text{, if } x < 1; \\ ax - b - 2 & \text{, if } 1 \le x \le 3; \\ \dfrac{x^2 - 9}{x - 3} & \text{, if } x > 3, \end{cases}$ where a and b are constants.

(a) Find the values of a and b that make f continuous everywhere.

For $x > 3$, $\dfrac{x^2-9}{x-3} = \dfrac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = x+3$

$\boxed{\text{I}}$ $\lim_{x \to 1^-} f(x) = b + 3a - 10 \overset{?}{=} a - b - 2 = \lim_{x \to 1^+} f(x)$

$\Rightarrow \quad 2b + 2a = 8$

$\Rightarrow \quad b = 4 - a$

$\boxed{\text{II}}$ $\lim_{x \to 3^-} f(x) = 3a - b - 2 \overset{?}{=} 6 = \lim_{x \to 3^+} f(x)$

$\Rightarrow \quad 3a - b = 8$

COMBINE $\boxed{\text{I}}$ & $\boxed{\text{II}}$ $\Rightarrow \quad 3a - (4-a) \overset{?}{=} 8$

$4a = 12$

$a = 3$

$b = 4 - a = 1$

$$\boxed{\begin{matrix} a = 3 \\ b = 1 \end{matrix}}$$

(b) Using the values for a and b from part (a), is the function $f(x)$ differentiable at $x = 1$? Clearly say NO or YES, and explain your answer in words. (Hint: Use all our derivative rules to analyze $f'(x)$ near $x = 1$.)

For $x < 1$, $f(x) = x^2 + 9x - 10$

$f'(x) = 2x + 9 \Rightarrow \lim_{x \to 1^-} f'(x) = 2(1) + 9 = 11$

For $1 < x < 3$, $f(x) = 3x - 1 - 2 = 3x - 3$

$f'(x) = 3 \Rightarrow \lim_{x \to 1^+} f'(x) = 3$

CONTINUOUS,
BUT SLOPES DON'T MATCH!

$\boxed{\text{NO}}$ NOT differentiable at $x = 1$.

5. (11 pts) A water balloon is thrown upward from a dorm window, it goes up for a bit then ultimately falls down to the ground and *coincidentally* lands near your math instructor. The height of the balloon is given by $s(t) = 80 - 16t^2 + 8t$ where t is in seconds since it was thrown and $s(t)$ is in feet.

(a) What is the average velocity of the balloon from $t = 0$ to $t = 2$ seconds? (include units)

$$\frac{s(2) - s(0)}{2 - 0} = \frac{(80 - 16(2)^2 + 8(2)) - (80)}{2} = \frac{-64 + 16}{2}$$

$$= \boxed{-24 \ \text{ft}/\text{sec}}$$

(b) Find and *completely simplify* the formula for the average velocity of the balloon from t to $t + h$. That is, find and completely simplify $\dfrac{s(t+h) - s(t)}{h}$.

$$\frac{[80 - 16(t+h)^2 + 8(t+h)] - [80 - 16t^2 + 8t]}{h}$$

$$= \frac{\cancel{80} - 16(t^2 + 2th + h^2) + 8t + 8h - \cancel{80} + \cancel{16t^2} - \cancel{8t}}{h}$$

$$= \frac{-32th - 16h^2 + 8h}{h}$$

$$= \boxed{-32t - 16h + 8}$$

(c) Find the instantaneous velocity at the time the balloon hits the ground? (include units)

$$s'(t) = -32t + 8$$

HIT GROUND $\Leftrightarrow$ $s(t) = 0$

$$80 - 16t^2 + 8t \stackrel{?}{=} 0$$

$$-2t^2 + t + 10 = 0$$

$$(-2t + 5)(t + 2) \stackrel{?}{=} 0$$

$t = 5/2$ or $t = -2$

$$s'\left(\tfrac{5}{2}\right) = -32\left(\tfrac{5}{2}\right) + 8$$

$$= -80 + 8$$

$$= \boxed{-72 \ \text{ft}/\text{sec}}$$

6. (12 pts) **NOTE: The two questions below are unrelated.**

(a) Find all points (a, b) at which the function $y = \dfrac{x^2}{3x-6}$ has a horizontal tangent line.

$$f'(x) = \frac{(3x-6)2x - 3x^2}{(3x-6)^2} \stackrel{?}{=} 0$$

$$6x^2 - 12x - 3x^2 \stackrel{?}{=} 0$$

$$x(3x - 12) \stackrel{?}{=} 0$$

$x = 0$ OR $x = 4$

$$y = \frac{(0)^2}{3(0)-6} = 0 \qquad y = \frac{(4)^2}{3(4)-6} = \frac{16}{6} = \frac{8}{3}$$

$$\boxed{(0,0) \ \& \ (4, 8/3)}$$

(b) Find all points (a, b) on the curve $y = \dfrac{5x}{3} + 4x^3 - 17$ where the tangent line at (a, b) also passes through the point $(0, 10)$.

$$y' = \frac{5}{3} + 12x^2$$

① (a,b) ON CURVE $\Rightarrow$ $b = \frac{5}{3}a + 4a^3 - 17$

② slope $= \frac{5}{3} + 12a^2$ AND slope $= \dfrac{b-10}{a-0}$

THUS, $\frac{5}{3} + 12a^2 \stackrel{?}{=} \dfrac{b-10}{a}$

$\Rightarrow$ $\frac{5}{3}a + 12a^3 = b - 10$

COMBINE ① & ② $\Rightarrow$ $\cancel{\frac{5}{3}a} + 12a^3 \stackrel{?}{=} \cancel{\frac{5}{3}a} + 4a^3 - 17 - 10$

$$8a^3 = -27$$

$$a^3 = -\frac{27}{8}$$

$$a = -\frac{3}{2}$$

$$y = \frac{5}{3}\left(-\frac{3}{2}\right) + 4\left(-\frac{3}{2}\right)^3 - 17$$

$$= -5/2 - \frac{27}{2} - 17 = -16 - 17$$

$$= -33$$

$$\boxed{\left(-\frac{3}{2}, -33\right)}$$