1. (a) (6 pts) Find the equation for the tangent line to the curve $y = \sqrt{e^{\sin(x)} + \ln(5x+1) + 1}$ at $x = 0$.

[redacted]

$$y' = \frac{e^{\sin(x)}\cos(x) + \frac{5}{5x+1}}{2\sqrt{e^{\sin(x)} + \ln(5x+1) + 1}}$$

[redacted]

$$y'(0) = \frac{1+5}{2\sqrt{1+0+1}} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3}{2}\sqrt{2}$$

$$y(0) = \sqrt{2}$$

$$\boxed{y = \frac{3}{\sqrt{2}}(x-0) + \sqrt{2} = \frac{3}{\sqrt{2}}x + \sqrt{2} = \frac{3}{2}\sqrt{2}x + \sqrt{2}}$$

(b) (6 pts) At $x = 0.3$ there is only one corresponding y value on the curve implicitly defined by

$$y^5 - x = yx^2 + 1.$$

Use the tangent line approximation at the point $(0, 1)$ to estimate the value of y that corresponds to $x = 0.3$ on this curve.

[redacted]

$$5y^4y' - 1 = y'x^2 + 2xy$$

$$5y^4y' - y'x^2 = 1 + 2xy$$

$$y' = \frac{1+2xy}{5y^4 - x^2} \qquad y'\big|_{(0,1)} = \frac{1+2(0)(1)}{5(1)^4 - (0)^2} = 1/5$$

[redacted]

$$y = \frac{1}{5}(x-0) + 1$$

$$y(0.3) = \frac{1}{5}(0.3) + 1 = (0.2)(0.3) + 1 = \boxed{1.06}$$

2. (a) (5 pts) Let $f(x)$ be a function such that it's derivative satisfies $2 \leq f'(x) \leq 5$ for all real values of x. Assuming $f(0) = 1$ and x is positive, by correctly stating and using the mean value theorem on the interval $[0, x]$ give an upper and lower bound on $f(x)$. (Note: Your bounds will be in terms of x).

[redacted] By the MVT, there exists a number c in $(0, x)$ such that

$$f(x) - f(0) = f'(c)(x - 0) = f'(c)x$$

[redacted] Since $2 \leq f'(c) \leq 5$ and x is positive, $2x \leq f(x) - f(0) \leq 5x$

And since $f(0) = 1$, $\boxed{2x + 1 \leq f(x) \leq 5x + 1}$

(b) (8 pts) Find and classify all critical numbers for $f(x) = \tan^{-1}(x^2) - \frac{1}{8}\ln(x^4 + 1)$.

[redacted]

$$f'(x) = \frac{2x}{1+x^4} - \frac{1}{8}\frac{4x^3}{(x^4+1)} = \frac{2x - \frac{1}{2}x^3}{1+x^4} = \frac{\frac{1}{2}x(4-x^2)}{1+x^4}$$

$$f'(x) \stackrel{?}{=} 0 \iff \tfrac{1}{2}x(4-x^2) = 0 \iff x = 0 \text{ or } x = -2 \text{ or } x = 2$$

OPTION 1

1^{st} derivative test:

$$f'(x) = \frac{\frac{1}{2}x(4-x^2)}{1+x^4}$$

$x < -2$	$-2 < x < 0$	$0 < x < 2$	$x > 2$
$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
$\frac{(-)(-)}{(+)}$	$\frac{(-)(+)}{(+)}$	$\frac{(+)(+)}{(+)}$	$\frac{(+)(-)}{(+)}$
↗	↘	↗	↘

[redacted]

OPTION 2

2^{nd} derivative test:

$$f''(x) = \frac{(1+x^4)(2 - \frac{3}{2}x^2) - 4x^3(2x - \frac{1}{2}x^3)}{(1+x^4)^2}$$

$f''(-2) \approx -0.235294116 < 0$ ∩

$f''(2) \approx -0.235294116 < 0$ ∩

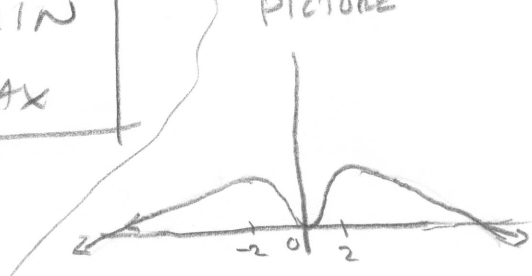
$f''(0) = 2 > 0$ ∪

[redacted] Either way:

$x = -2$ gives a local MAX
$x = 0$ gives a local MIN
$x = 2$ gives a local MAX

AS IDE

PICTURE



3. (a) (6 pts) Consider the function $f(x) = \dfrac{e^x}{ax^2+b}$ where a and b are positive constants. Find general conditions on a and b under which $f(x)$ will have two (real) critical numbers and find these two critical numbers.

$$f'(x) = \frac{(ax^2+b)e^x - 2ax\,e^x}{(ax^2+b)^2} \stackrel{?}{=} 0$$

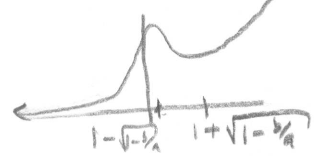
NEEDS TO BE POSITIVE

$$\Rightarrow e^x(ax^2 - 2ax + b) = 0 \Rightarrow x = \frac{2a \pm \sqrt{4a^2 - 4ab}}{2a}$$

So $\boxed{a > b}$

$$\boxed{x = \frac{2a}{2a} \pm \frac{1}{2a}\sqrt{4a^2-4ab} = 1 \pm \frac{1}{a}\sqrt{a^2-ab} = 1 \pm \sqrt{1 - b/a}}$$

ASIDE: PICTURE



(b) (7 pts) Find the absolute max and min values of $f(x) = x^{(-1/x^2)}$ on the interval $[\frac{1}{2}, 2]$.

$$\ln(y) = -\frac{1}{x^2}\ln(x)$$

$$\frac{1}{y}y' = \frac{2}{x^3}\ln(x) - \frac{1}{x^3}$$

$$y' = x^{(-1/x^2)}\frac{1}{x^3}(2\ln(x) - 1) \stackrel{?}{=} 0 \quad \text{on } [\tfrac{1}{2}, 1)$$

ONLY WHEN $2\ln(x) - 1 = 0 \Rightarrow \ln(x) = 1/2 \Rightarrow x = e^{1/2} = \sqrt{e}$

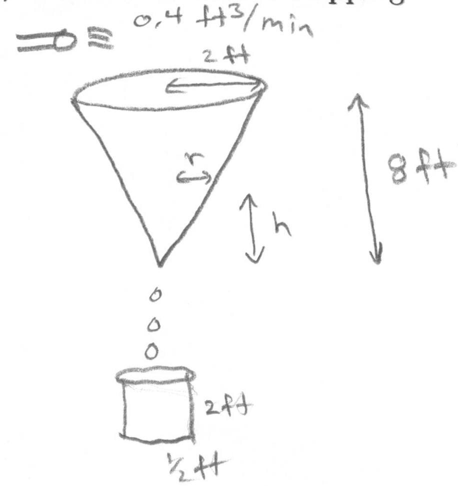
$$f(\tfrac{1}{2}) = (\tfrac{1}{2})^{-4} = 2^4 = \boxed{16} \longleftarrow \text{ABS. MAX}$$

$$f(e^{1/2}) = (e^{1/2})^{-1/e} = \boxed{e^{-1/2e}} \approx 0.8319859539 \longleftarrow \text{ABS MIN}$$

$$f(1) = 2^{-1/4} \approx 0.6408964153$$

4. At time $t = 0$ min, you start pumping milk into a cone [redacted] at a constant rate of 0.4 ft^3/min. The bottom point of the cone is dripping, so that the cone is also losing volume at a constant, but unknown, rate, of c ft^3/min. The cone is 8 feet high with a radius of 2 feet at the top. Recall: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

(a) (7 pts) At some particular time you measure the height of the milk in the cone to be 2 feet with the height increasing at a rate of $\frac{1}{3}$ ft/min. Find the rate, c, at which the milk is dripping out of the funnel.



$$\frac{h}{r} = \frac{8}{2} \Rightarrow h = 4r \Rightarrow \frac{dh}{dt} = 4\frac{dr}{dt}$$

$$V = \frac{1}{3}\pi r^2(4r) = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$h = 2 \Rightarrow r = \tfrac{1}{2}$$

$$\frac{dh}{dt} = \frac{1}{3} \Rightarrow \frac{dr}{dt} = \frac{1}{12}$$

Thus, $\frac{dV}{dt} = 4\pi\left(\frac{1}{2}\right)^2 \frac{1}{12} = \pi/12$

Volume is coming in at $0{,}4 \frac{ft^3}{min}$ and leaving at c ft³/min

So $0{,}4 - c = \pi/12$

Hence, $\boxed{c = 0{,}4 - \pi/12 \simeq 0{,}138200612 \text{ ft}^3/\text{min}}$

(b) (4 pts) The milk drips from the funnel into a cylindrical bucket at the constant rate you found in part (a). The cylindrical bucket is 2 feet high and has a radius of 1/2 foot. At what time, t, will the bucket be full?
(This is when you plan to dump the bucket of milk on Dr. Loveless' head).

$V = \pi r^2 h$ $\Rightarrow$ TOTAL BUCKET VOLUME $= \pi\left(\frac{1}{2}\right)^2 \cdot 2 = \pi/2$ ft³

RATE INTO BUCKET $= 0{,}4 - \pi/12$ ft³/min

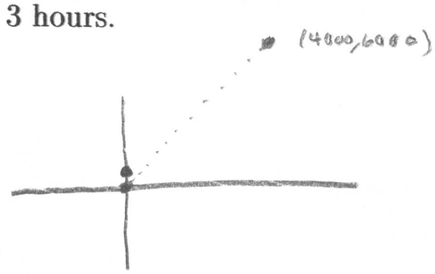
BUCKET FULL WHEN $\boxed{t = \dfrac{\pi/2}{0{,}4 - \pi/12} = \dfrac{6\pi}{4{,}8 - \pi} \text{ min}}$

$\simeq 11.3660591$ min

ASIDE AT THIS TIME, THE BUCKET WILL BE FULL.
THE CONE WILL HAVE VOLUME $\frac{\pi}{12}\frac{ft^3}{min} \cdot 11.3660591 min = 2.9756$ ft³
SO THE CONE WON'T BE FULL.

5. Since you know that Dr. Loveless has motion sickness, you and some classmates tie him to the edge of a merry-go-round which happens to be on a moving train. The merry-go-round has radius of 4 feet and is rotating at a counterclockwise constant rate of 5 revolutions per hour. At time $t = 0$, Dr. Loveless is on the northernmost edge of the merry-go-round.

The train moves at a constant speed in such a way that the **center** of the merry-go-round is at the origin at time $t = 0$ and at the point $(4000, 6000)$ at time $t = 3$ hours.



The model for this motion is

$$x(t) = at + 4\cos(\theta_0 + wt)$$
$$y(t) = bt + 4\sin(\theta_0 + wt)$$

(a) (4 pts) Find the constants θ_0, w, a, and b.

Recall: General circular motion $x(t) = x_c + r\cos(\theta_0 + wt)$

$y(t) = y_c + r\sin(\theta_0 + wt)$

where $(x_c, y_c) =$ center

$\boxed{\theta_0 = \pi/2}$ $\boxed{w = \frac{5\text{ rev}}{\text{hr}} = \frac{10\pi\text{ rad}}{\text{hr}}}$

$x_c = at \Rightarrow 4000 = 3a$

$y_c = bt \Rightarrow 6000 = 3b$

$\boxed{a = \frac{4000}{3}}$ $\boxed{b = \frac{6000}{3} = 2000}$

(b) (7 pts) At $t = 4.5$ hours, Dr. Loveless comes untied and falls off the merry-go-round. Find the equation for the tangent line path he follows at $t = 4.5$ hours.

$x\left(\frac{9}{2}\right) = \frac{4000}{3}\cdot\frac{9}{2} + 4\cos\left(\frac{\pi}{2} + 10\pi\,\frac{9}{2}\right) = 6000 + 0 = 6000$

$y\left(\frac{9}{2}\right) = 2000\cdot\frac{9}{2} + 4\sin\left(\frac{\pi}{2} + 10\pi\,\frac{9}{2}\right) = 9000 - 4 = 8996$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2000 + 40\pi\cos\left(\frac{\pi}{2} + 10\pi t\right)}{4000/3 - 40\pi\sin\left(\frac{\pi}{2} + 10\pi t\right)}\Bigg|_{t=9/2} = \frac{2000 + 0}{\frac{4000}{3} + 40\pi}$

$= \frac{6000}{4000 + 120\pi} = \frac{150}{100 + 3\pi}$

≈ 1.370804701

$$\boxed{\begin{array}{l} y = \frac{150}{100 + 3\pi}(x - 6000) + 8996 \\ y = 1.370804701(x - 6000) + 8996 \end{array}}$$