1. (12 pts) In each part, find $\frac{dy}{dx}$. Simplify your answers.

(a) $y = \ln(1 + x^4) - \tan^{-1}(x^2)$

$$\frac{dy}{dx} = \frac{1}{1+x^4} \cdot 4x^3 - \frac{1}{1+(x^2)^2} \cdot 2x$$

$$\boxed{\frac{dy}{dx} = \frac{4x^3 - 2x}{1+x^4} = \frac{2x(2x^2-1)}{1+x^4}}$$

(b) $y^3 = (6x)^{(x^2)}$ (put your answer in terms of x)

$$3\ln(y) = x^2 \ln(6x)$$

$$\frac{3}{y}\frac{dy}{dx} = x^2 \cdot \frac{1}{6x} \cdot 6 + 2x\ln(6x)$$

$$\frac{dy}{dx} = \frac{1}{3} y(x + 2x\ln(6x)) = \frac{1}{3}(6x)^{x^2/3}(x + 2x\ln(6x))$$

$$\boxed{\frac{dy}{dx} = \frac{1}{3} x (6x)^{x^2/3}(1 + 2\ln(6x))}$$

(c) $x(t) = t\cos(t)$, $y(t) = e^t - t$ (your answer will be in terms of t)

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t - 1}{\cos(t) - t\sin(t)}}$$

2. (7 pts) Use implicit differentiation to find the equation of the tangent line to the curve

$$x^2 - xy^2 + 1 = (x + y^2)^2$$

at the point $(x, y) = (0, 1)$.

$$2x - 2xy\frac{dy}{dx} - y^2 = 2(x+y^2)\left(1 + 2y\frac{dy}{dx}\right)$$

$x=0, y=1 \Rightarrow \quad 2(0) - 2(0)(1)\frac{dy}{dx} - (1)^2 = 2(0+1^2)\left(1+2(1)\frac{dy}{dx}\right)$

$\Rightarrow \quad -1 = 2\left(1+2\frac{dy}{dx}\right)$

$-1 = 2 + 4\frac{dy}{dx}$

$-3 = 4\,dy/dx \qquad \frac{dy}{dx} = -3/4$ at $(x,y)=(0,1)$

LINE: $\boxed{y = -\frac{3}{4}(x-0)+1 = -\frac{3}{4}x+1}$

3. (7 pts) Find the absolute maximum and absolute maximum values of $g(x) = 14x^2 - x^4$ on the interval $[-2, 4]$. Justify your answers.

NOTE: DEFINED EVERYWHERE

CRITICAL NUMBERS

$$\overbrace{g'(x) = 28x-4x^3}^{\text{defined everywhere}} \stackrel{?}{=} 0$$

$$4x(7-x^2) = 0$$

$$x = 0, \quad \underbrace{x=-\sqrt{7}}_{\text{NOT IN INTERVAL}}, \quad x=\sqrt{7}$$

EVALUATE

critical #'s:
$g(0) = 14(0)^2 - (0)^4 = 0$

$g(\sqrt{7}) = 14(\sqrt{7})^2 - (\sqrt{7})^4 = \boxed{49 = \text{ABS. MAX}}$

endpoints:
$g(-2) = 14(-2)^2 - (-2)^4 = 40$

$g(4) = 14(4)^2 - (4)^4 = \boxed{-32 = \text{ABS. MIN}}$

NOTE, THIS IS EVEN SO IT IS SYMMETRIC ABOUT THE y-axis)

4. (12 points) Consider the function $f(x) = 6x^{4/3} - x^2$. Justify your work in each part using appropriate first and/or second derivative tests.

(a) Find all critical points of $f(x)$. Classify each critical point as a local max, local min, or neither.

(NOTE: DEFINED EVERYWHERE)

$$f'(x) = 8x^{1/3} - 2x \overset{?}{=} 0 \Rightarrow 4x^{1/3} - x = 0$$

$$4x^{1/3} = x$$

$x = 0$ or $\rightarrow 4 = x^{2/3}$

$$x = \pm 4^{3/2} = \pm 8$$

$\nearrow$ $\searrow$ $\nearrow$ $\searrow$

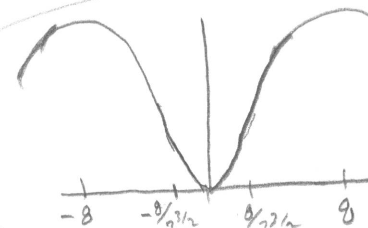
$f' > 0$ | -8 | $f' < 0$ | 0 | $f' > 0$ | 8 | $f' < 0$

$f'(-100) = 162.867$ $\quad f'(-1) = -6$ $\quad f'(1) = 6$ $\quad f'(100) = -162.86$

$x = -8$ gives a local max
$x = 0$ gives a local min
$x = 8$ gives a local max

1st deriv. test

ASIDE Here's the graph



UNDEFINED AT $x = 0$

(b) Find all inflection points of $f(x)$.

$$f''(x) = \frac{8}{3}x^{-2/3} - 2 = \frac{8}{3x^{2/3}} - 2$$

$$\frac{8}{3x^{2/3}} - 2 = 0 \Rightarrow \frac{8}{3x^{2/3}} = 2 \Rightarrow 8 = 6x^{2/3}$$

$$\frac{4}{3} = x^{2/3} \Rightarrow x = \pm\left(\frac{4}{3}\right)^{3/2} = \pm\frac{8}{3^{3/2}} \approx \pm 1.5396007$$

Concave down | Concave up | Concave up | Concave down.

$f'' < 0$ | $-\left(\frac{4}{3}\right)^{3/2}$ | $f'' > 0$ | 0 | $f'' > 0$ | $\left(\frac{4}{3}\right)^{3/2}$ | $f'' < 0$

$f''(-10) = -1.425$ $\quad f''(-1) = 2/3$ $\quad f''(1) = 2/3$ $\quad f''(10) = -1.425$

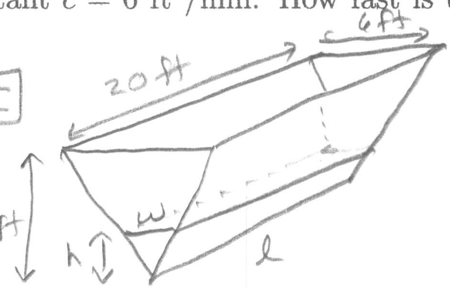
$x = -\frac{8}{3^{3/2}}$ and $x = \frac{8}{3^{3/2}}$ are the only inflection pts.

5. (10 pts) A trough is 20 ft long and its ends have the shape of isosceles triangles that are 6 feet across at the top and have a height of 2 feet.

The trough is placed under a pipe which is leaking out water at a constant rate of c ft^3/min.

(a) Assume it is known that the water is leaking at a constant $c = 6$ ft^3/min. How fast is the water level rising when the water is 9 inches deep?

$\boxed{\text{I}}$



Similar triangles : $\frac{w}{h} = \frac{6}{2} \Rightarrow w = 3h$

$\boxed{\text{II}}$ $V = \frac{1}{2} w h l = \frac{1}{2} 3h \cdot h \cdot 20$

$V = 30h^2$

$\boxed{\text{III}}$ $\frac{dV}{dt} = 60h \frac{dh}{dt}$

$\boxed{\text{IV}}$ $\frac{dV}{dt} = 6$ ft^3/min , $h = 9\text{in} = 0.75$ ft $\Rightarrow$ $6 = \underbrace{60 \cdot 0.75}_{45} \cdot \frac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = \frac{6}{45} = \frac{2}{15} \text{ ft/min} = 0.1\bar{3} \text{ ft/min} = 1.6 \text{ in/min}}$$

(b) Assume c is not known initially, but it is known to be constant. At time $t = 0$, the trough is empty. Two minutes later, the trough is 6 inches (0.5 feet) deep. Find the constant rate at which the volume of water is leaking, *i.e.* find c.

$V(0) = 0$ ft^3

$t = 2$ min $\Rightarrow h = \frac{1}{2}$ ft $\Rightarrow V = 30\left(\frac{1}{2}\right)^2 = 7.5$ ft^3

So $V(2) = 7.5$ ft^3

AVERAGE RATE $= \frac{V(2) - V(0)}{2 - 0} = \frac{7.5 - 0}{2 - 0} = 3.75$ ft^3/min

Since $\frac{dV}{dt}$ is a constant, the average rate is the same as the instantaneous rate, so

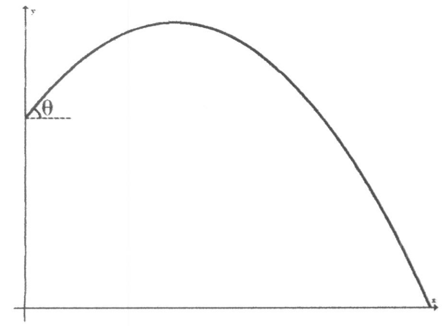
$$\boxed{\frac{dV}{dt} = c = 3.75 \text{ ft}^3\text{/min}}$$

6. (12 pts)

A pumpkin is fired from a cannon off a cliff and into a corn field. The location of the pumpkin at time t is given by the parametric equations

$$x(t) = 50\cos(\theta)t \quad \text{and} \quad y(t) = 30 + 50\sin(\theta)t - 16t^2,$$

where the angle, θ, is the initial angle at which the pumpkin is fired measured from the horizontal. All distances are in feet and time is in seconds.



(a) If $\theta = \frac{\pi}{4}$ radians, find the time(s) when the horizontal velocity is twice the size of the vertical velocity.

$x(t) = 25\sqrt{2}\,t,\ y(t) = 30 + 25\sqrt{2}\,t - 16t^2$

$x'(t) = 25\sqrt{2},\ y'(t) = 25\sqrt{2} - 32t$

WANT $\quad 25\sqrt{2} = 2(25\sqrt{2} - 32t)$

$25\sqrt{2} = 50\sqrt{2} - 64t$

$-25\sqrt{2} = -64t$

$\boxed{t = \frac{25\sqrt{2}}{64} \text{ sec} \approx 0.552427 \text{ sec}}$

(b) If we wanted to find an angle, θ, in order to make the pumpkin land on a target at (100,0), we would ultimately need to solve the equation (you don't have to derive this):

$$15 + 50\tan(\theta) - 32\sec^2(\theta) = 0.$$

There are two answer between 0 and pi/2 radians and one of the answers is 'near' $\theta = \pi/4$. Find the linear approximation of $f(\theta) = 15 + 50\tan(\theta) - 32\sec^2(\theta)$ at $\theta = \pi/4$. Use the linear approximation to estimate a solution to $f(\theta) = 0$.

$f(\pi/4) = 15 + 50\underbrace{\tan(\pi/4)}_{=1} - 32\underbrace{\sec^2(\pi/4)}_{\sqrt{2}} = 15 + 50(1) - 32(\sqrt{2})^2$
$= 65 - 64 = 1$

$f'(\theta) = 50\sec^2(\theta) - 64\sec(\theta)\sec(\theta)\tan(\theta) = \sec^2(\theta)(50 - 64\tan(\theta))$

$f'(\pi/4) = (\sqrt{2})^2(50 - 64(1)) = 2(-14) = -28$

$f(\theta) \approx f(\pi/4) + f'(\pi/4)(\theta - \pi/4)$ for $\theta \approx \pi/4$

$= 1 - 28(\theta - \pi/4)$

$f(\theta) = 0$ approx. when $1 - 28(\theta - \pi/4) = 0$ near $\theta = \pi/4$

$\theta - \pi/4 = 1/28$

$\boxed{\theta = \pi/4 + 1/28 = 0.8211124491 \text{ radians} = 47.04627764 \text{ degrees}}$

(BONUS POINT) One extra credit bonus point if you can give an exact form answer for both angles θ between 0 and $\pi/2$ that solve this equation (put your answer on the back of this page).

$\boxed{\text{Where the equation came from}}$

$(x,y) = (100,0)$ implies

① $100 = 50\cos(\theta)t$

② $0 = 30 + 50\sin(\theta)t - 16t^2$

Solving ① for t gives $t = \frac{2}{\cos(\theta)} = 2\sec(\theta)$

Substituting in ② gives

$$0 = 30 + 50\sin(\theta)\frac{2}{\cos(\theta)} - 16\left(\frac{2}{\cos(\theta)}\right)^2$$

simplifying

$$0 = 30 + 100\tan(\theta) - 64\sec^2(\theta)$$

dividing by 2

$$\boxed{0 = 15 + 50\tan(\theta) - 32\sec^2(\theta)}$$

$\boxed{\text{BONUS}}$ $\sec^2(\theta) = 1 + \tan^2(\theta)$ ← Identity mentioned in class

So

$$15 + 50\tan(\theta) - 32(1 + \tan^2(\theta)) = 0$$

expand

$$15 + 50\tan(\theta) - 32 - 32\tan^2(\theta) = 0$$

rearrange flipping signs

$$32(\tan(\theta))^2 - 50\tan(\theta) + 17 = 0$$

This is a quadratic equation in $\tan(\theta)$, so the quadratic formula gives

$$\tan(\theta) = \frac{50 \pm \sqrt{50^2 - 4(32)(17)}}{2(32)}$$

$$\tan(\theta) = \frac{50 \pm \sqrt{324}}{64} \leftarrow 18$$

$\tan(\theta) = \frac{68}{64} = \frac{17}{16}$ OR $\tan(\theta) = \frac{32}{64} = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{17}{16}\right)$ OR $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

≈ 0.8156919 rad ≈ 0.463647 rad

$= 46.73570459$ deg. $= 26.56505118$ deg

Our linear approximation estimate was off by 0.3 degrees from the actual answer.