Parametric Equations

Typical, high school pre-calculus and algebra courses only discuss parametric equations lightly and focus on the fundamental functions (polynomials, exponentials, trig, *etc.*) and this is a perfectly reasonable approach. However, when it comes time to use our mathematical toolbox on real problems parametric equations naturally arise. Particularly when the we encounter motion for which the location is a function of time. For many of these scenarios, it is easier and much more useful to have the coordinates x and y given as separate functions of time (and this will be even more useful as we go to 3 dimensions in Math 126). The basic facts we have so far (after our lecture on 10.2) are as follows:

1. If $x = x(t)$ and $y = y(t)$, then we can graph the parametric curve in the (x, y) plane by:

   - Selecting various values of t and calculating x and y. Then plot these points (x, y) and indicate the direction of increasing time.

   - Attempting to eliminate the parameter by either solving for t in one equation and using this to replace t in the other equation or using some identity (likely the trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$) to combine the equations and eliminate the parameter. If we in fact eliminate the parameter to get an equation in terms of x and y, then that equation represents the path of curve (but this equation doesn't contain any time information so we still have to go back to the parametric equations to plot some points and indicate direction).

2. $x'(t) = \frac{dx}{dt} =$ 'horizontal velocity at time t' = 'instantaneous rate of change of x with respect to t'.

3. $y'(t) = \frac{dy}{dt} =$ 'vertical velocity at time t' = 'instantaneous rate of change of y with respect to t'.

4. If we want to know about the rate of change of y with respect to x, we used the chain rule to verify that

$$\frac{dy}{dx}(t) = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \text{ the slope of the tangent to the } \textit{path} \text{ at time } t$$

5. If we want to know the second derivative of y with respect to x, we have to be extra careful as we discussed and use the correct formula in terms of t:

$$\frac{d^2y}{dx^2}(t) = \frac{\frac{d}{dt}\left[\frac{dy}{dx}(t)\right]}{dx/dt} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}(t)\right]}{x'(t)}.$$

Essentially, this gives you more options in a problem involving parametric equations. You can either eliminate the parameter and do everything in terms of x and y like we have been doing all quarter. Or you can keep everything in terms of t and work from there (at any point if you have t and you want x and y, then you can just go back to the given parametric equation). In some cases, you must use one method or the other (perhaps because it is not easy to eliminate the parameter).

You have already explored several situations in the homework were parametric equations are useful including:

- Uniform Linear Motion (motion on a straight line at a constant speed):

$$x(t) = a + bt \text{ and } y(t) = c + dt.$$

- Uniform Circular Motion (motion on a circle at a constant speed):

$$x(t) = r\cos(\theta_0 + \omega t) \text{ and } y(t) = r\sin(\theta_0 + \omega t).$$

We get elliptical motion if each equation has a different value for r.

- Projectile Motion without air resistance (this was the cyclist off the ramp problem):

$$x(t) = x_0 + v_0 \cos(A)t \text{ and } y(t) = y_0 + v_0 \sin(A)t - \frac{g}{2}t^2.$$

A Bit About Parametric Equation Graphs

Recall: A **function**, $y = f(x)$, has only one y value for each x value. In other words, it passes the vertical line test. Function can look quite exotic, but they cannot have two y's for one x. I hope we have a basic understanding of what graphs of all the fundamental pre-calculus functions look like.

For parametric equations, the graphs (even from basic functions) can be quite exotic. Since x and y are given separately, the graph depends on the interplay between the functions defining x and y. Parametric curves often don't pass the vertical line test in the xy-plane, which also adds a new wrinkle to what the graphs can look like. In any case, if we can't quickly eliminate the parameter, then we typically will have to plot a lot of points to get a full idea of what the graph looks like (or we will have to do a careful analysis of what is happening in terms of x and y separately).
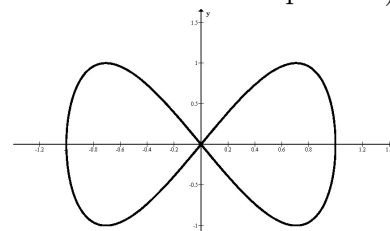
Here are some examples of more exotic parametric curves just to give you an idea of what can happen (especially in the case where x and y are defined in terms of sinusoidal functions with different periods).
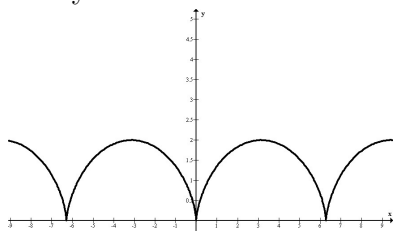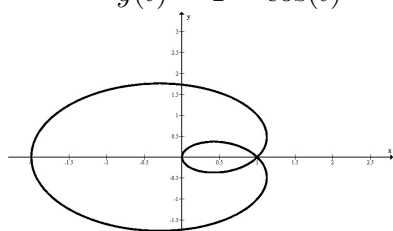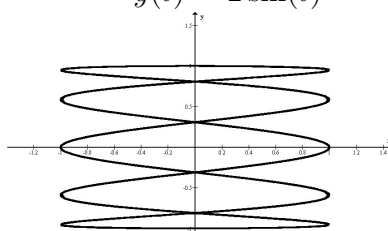

Cycloid: $x(t) = t - \sin(t)$
$y(t) = 1 - \cos(t)$
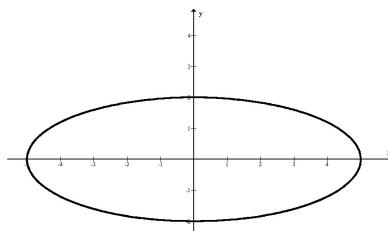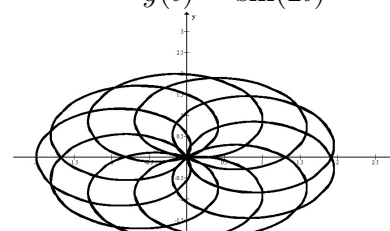
Ellipse: $x(t) = 5\cos(t)$
$y(t) = 2\sin(t)$

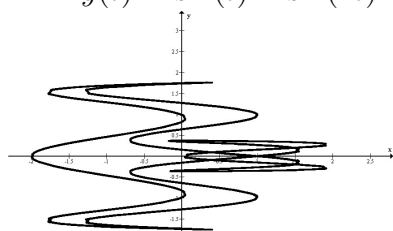$x(t) = \cos(t)$
$y(t) = \sin(2t)$

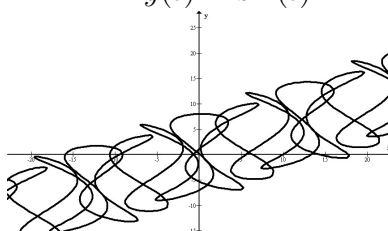$x(t) = \cos(t) - \cos(2t)$
$y(t) = \sin(t) - \sin(2t)$

$x(t) = \cos(5t)$
$y(t) = \sin(t)$

$x(t) = \cos(t) - \cos(10t)$
$y(t) = \sin(t) - \sin(10t)$

$x(t) = \cos(t) - \cos(10t)$
$y(t) = \sin(t) - \sin(2t)$

$x(t) = 2t + 3\sin(7t)$
$y(t) = t + 8\cos(3t)$

It might be fun to use a graphing calculator, or online parametric grapher, to play around with different functions you can get. In graphics, parametric equations are useful as they are a good way to graph various pictures (and they are more flexible and easier to use than the typical $y = f(x)$ if you want to generate a particular picture).