3.10 Linear Approximation Review

1. Honestly, there are no new techniques in this section. But it is a very important conceptual section as it explains an important and useful application of tangent lines.

2. Recall: The equation for the tangent line to $f(x)$ at $x = a$ is given by

$$y = f'(a)(x - a) + f(a) \text{ which, of course, can also be written as } y = f(a) + f'(a)(x - a).$$

   The tangent line has two important properties compared with the function $f(x)$ at $x = a$:

   - They have the same value at $x = a$ (if you plug $x = a$ into the tangent line or the function you get the same output because the tangent line touches the graph of $f(x)$ at $x = a$).
   - They have the same slope at $x = a$ (in fact that is the definition of the tangent line).

   In this sense, the tangent line is the 'best' approximation of the function $f(x)$ near $x = a$. Thus, "near" $x = a$ the tangent line and the function $f(x)$ are going to be close together.

3. When we are thinking of the tangent line in this way, we say $y = f(a) + f'(a)(x - a)$ or $L(x) = f(a) + f'(a)(x - a)$ is the **linear approximation** (or tangent line approximation or linearization) to $f(x)$ at $x = a$. We can then use this approximation to estimate values for the original function.

4. Here are some of the examples we found in class (you should certainly know how we got these, but these are just examples, you need to know the method):

   (a) $\sqrt{x} \approx 9 + \frac{1}{18}(x - 81)$ for $x \approx 81$.
   (b) $x^{1/3} \approx 2 + \frac{1}{12}(x - 8)$ for $x \approx 8$.
   (c) $\sin(x) \approx x$ for $x \approx 0$.
   (d) $\sin(x) \approx 1$ for $x \approx \frac{\pi}{2}$.
   (e) $e^x - 4x \approx 1 - 3x$ for $x \approx 0$.

   Questions about the error involved in using such an approximation will be discussed at the end of Math 126 when we talk about Taylor Series. For now, we will just use some common sense that if you stray too far from the value $x = a$, the approximation will lose accuracy.

5. Applications:

   (a) Evaluating functions on a desert island: If we want to know the value of $7.8^{1/3}$, we can find the linear approximation at an 'easy' to evaluate point nearby (in this case $x = 8$). From above we found the tangent line at $x = 8$ to be $y = 2 + \frac{1}{12}(x - 8)$. Since 7.8 is near 8, we can say

$$7.8^{1/3} \approx 2 + \frac{1}{12}(7.8 - 8) = 2 + \frac{1}{12}(-0.2) = 2 - \frac{1}{60} = \frac{119}{60} = 1.983333$$

   So without a calculator (you do have do a long division by hand at the end) you can accurately approximate the value of $7.8^{1/3}$. (The actual value to several digits is 1.983192483).

   (b) Simplifying problems: Important applications of this idea arise in physics, upper level math courses, and other sciences. When we encounter something that is difficult to work with, solve or simplify, we can use the linear approximation instead. Provided we can show that the error in our approximation is minimal, this essentially can replace a very complicated problem by a simpler problem. I mentioned the pendulum in lecture and how the acceleration involves the function $-9.8\sin(\theta)$ (which can be messy to study). For small angles $\theta \approx 0$, we can try to study what is going on by replacing $\sin(\theta)$ by it's linear approximation which from above is just $\sin(\theta) \approx \theta$, so the acceleration now involves the function -9.8θ (which turns out to be easier to work with). This idea of using approximations is widespread in the sciences and it is an important and useful idea.

(c) Solving equations (**Newton's Method**): If an equation is difficult (or impossible) to solve with symbolic methods, then we can replace a messy function with it's linear approximation and solve the equation with the linear approximation instead. This doesn't always give a good answer because the solution may not be near the location of the tangent line. However, if we start with a value near the solution, then using the tangent line approximation makes sense. And even better once we use it once we will have a number closer to the correct solutions, and then we can use it again to get even closer (and again and again if we wish. Here is Newton's Method (to approximation the solution of $f(x) = 0$):

i. Make a good guess of the answer. Call it $x = a$.
ii. Find the linear approximation to $f(x)$ at $x = a$, which would look like $y = f(a) + f'(a)(x - a)$.
iii. Instead of solving $f(x) = 0$, solve $f(a) + f'(a)(x - a) = 0$ which gives $x = a - \frac{f(a)}{f'(a)}$.
iv. If this answer is close enough to accurate for you, stop. If not start over with this as your new guess.

If you say $x_0 =$ guess zero, and $x_1 =$ the next guess you get, and $x_2 =$ the next guess and so on. You can formally write this method with the relationship $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. This method, or variants of it, can be used by your calculator to numerically solve equations.