1. (10 pts) (Remember, give simplified exact values in your final answers as indicated on cover)

(a) Find the slope of the tangent line to $y = \arctan(\sqrt{x})\ln(3x^2 - 1)$ at $x = 1$.

$$y' = \frac{1}{(1+(\sqrt{x})^2)} \frac{1}{2\sqrt{x}} \ln(3x^2-1) + \arctan(\sqrt{x}) \frac{6x}{3x^2-1}$$

$$y'(1) = \frac{1}{2} \cdot \frac{1}{2} \ln(2) + \arctan(1) \cdot \frac{6}{2}$$

$$= \frac{\ln(2)}{4} + \frac{\pi}{4} \cdot 3$$

$$= \boxed{\frac{\ln(2)}{4} + \frac{3\pi}{4} = \frac{1}{4}(\ln(2) + 3\pi)}$$

(b) Find $\frac{dy}{dx}$ for $y = x^{\sin(\pi x)}$.

$$y = e^{\sin(\pi x)\ln(x)}$$

$$\frac{dy}{dx} = e^{\sin(\pi x)\ln(x)} \left( \pi\cos(\pi x)\ln(x) + \frac{\sin(\pi x)}{x} \right)$$

$$= \boxed{x^{\sin(\pi x)} \left( \pi\cos(\pi x)\ln(x) + \frac{\sin(\pi x)}{x} \right)}$$

SAME

OR

$$\ln(y) = \sin(\pi x)\ln(x)$$

$$\Rightarrow \frac{dy}{dx} = \boxed{y\left(\pi\cos(\pi x)\ln(x) + \frac{\sin(\pi x)}{x}\right)}$$

2. (10 pts) Consider the curve implicitly defined by $x^2y^3 - \dfrac{1}{x+y^2} = 5$.

(a) The point $(x, y) = (-2, 1)$ is on the curve.
Find the equation for the tangent line to the curve at $(-2, 1)$.

$$2xy^3 + 3x^2y^2\frac{dy}{dx} + (x+y^2)^{-2}\left(1+2y\frac{dy}{dx}\right) = 0$$

$$\left.\begin{array}{l} x=-2 \\ y=1 \end{array}\right\} \Rightarrow -4 + 12\frac{dy}{dx} + \frac{1}{(-2+1)^2}\left(1+2\frac{dy}{dx}\right) = 0$$

$$\Rightarrow -4 + 12\frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 0$$

$$14\frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = 3/14$$

$$\boxed{y = \frac{3}{14}(x+2) + 1}$$

(b) Use the tangent line approximation at $(-2, 1)$ to approximate a $y$-coordinate on the curve when $x = -1.9$.

$$y = \frac{3}{14}(-1.9+2) + 1 = \boxed{\frac{3}{14}(0.1) + 1 \approx 1.0214}$$

3. (10 pts) At time $t$, the location of a particle is given by:

$$x = t^3 - t \quad , \quad y = 8te^{(-t/4)}$$

The path of the particle in the $xy$-plane is shown below in part (b).

(a) Find all times $t$ when the curve has a horizontal or vertical tangent.

i. Horizontal Tangent time(s):

$$\frac{dy}{dt} = 8e^{-\frac{1}{4}t} - 2te^{-\frac{1}{4}t} = (8-2t)e^{-\frac{1}{4}t} \overset{?}{=} 0$$

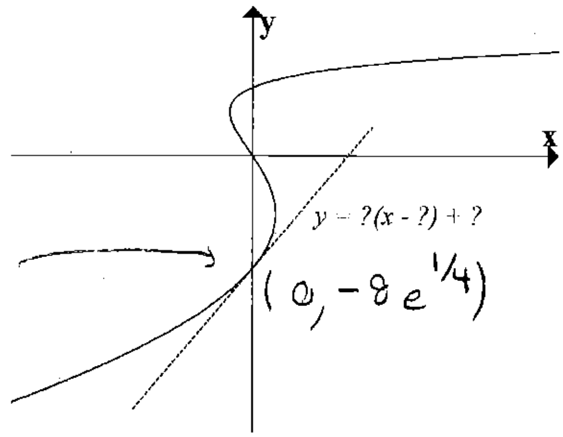
$$\boxed{t = 4}$$

ii. Vertical Tangent time(s):

$$\frac{dx}{dt} = 3t^2 - 1 \overset{?}{=} 0 \implies t^2 = \frac{1}{3} \qquad \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\boxed{t = \pm\sqrt{\frac{1}{3}}}$$

(b) Find the equation for the tangent line at the negative $y$-intercept. (shown in picture)




$x = 0$

$\Rightarrow 0 = t^3 - t = t(t^2 - 1)$

$0 = t(t-1)(t+1)$

$t = 0,\ t = 1,$ or $t = -1$

$(0, -8e^{1/4})$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(8+2)e^{1/4}}{(3-1)}$$

$$= \frac{10}{2}e^{1/4} = 5e^{1/4}$$

$$\boxed{y = 5e^{1/4}(x - 0) - 8e^{1/4}}$$

$$= 5e^{1/4}x - 8e^{1/4} = e^{1/4}(5x - 8)$$

4. (10 pts) Find the absolute max and min of $f(x) = \dfrac{x-2}{x^2-3x+3}$ on the interval $x = 2$ to $x = 10$. (Clearly, check all appropriate points and label your final answers).

$$f'(x) = \frac{(x^2-3x+3)(1) - (x-2)(2x-3)}{(x^2-3x+3)^2}$$

$$= \frac{x^2-3x+3-(2x^2-4x-3x+6)}{(x^2-3x+3)^2}$$

$$= \frac{x^2-3x+3-2x^2+7x-6}{(x^2-3x+3)^2} \stackrel{?}{=} 0$$

$$\Rightarrow \quad -x^2+4x-3 \stackrel{?}{=} 0$$

$$x^2-4x+3 = 0$$

$$(x-1)(x-3) = 0$$

$$\underbrace{x=1}_{\text{NOT IN GIVEN DOMAIN}} \quad \text{or} \quad x = 3 \quad \left\{ \text{CRITICAL NUMBERS} \right.$$

$$f(2) = \frac{2-2}{(2)^2-3(2)+3} = \boxed{0 \leftarrow \text{ABS. MIN}}$$

$$f(3) = \frac{3-2}{3^2-3(3)+3} = \boxed{\frac{1}{3} \leftarrow \text{ABS. MAX}}$$

$$f(10) = \frac{10-2}{10^2-3(10)+3} = \frac{8}{73}$$

5. (10 pts) A baseball diamond is a square with sides of length 90 ft. Kyle hits the ball and runs toward first base.

At the instant Kyle has run 30 feet, he is running at 24 ft/sec. 40 ft from home,

At this same instant, there is another runner, Nelson, [redacted] between 3rd base and homeplate and he is running 20 ft/sec toward home.

At what rate is the distance between Nelson and Kyle changing at this instant?
Appropriately indicate if this distance is increasing (positive rate) or decreasing (negative rate). (also give units).

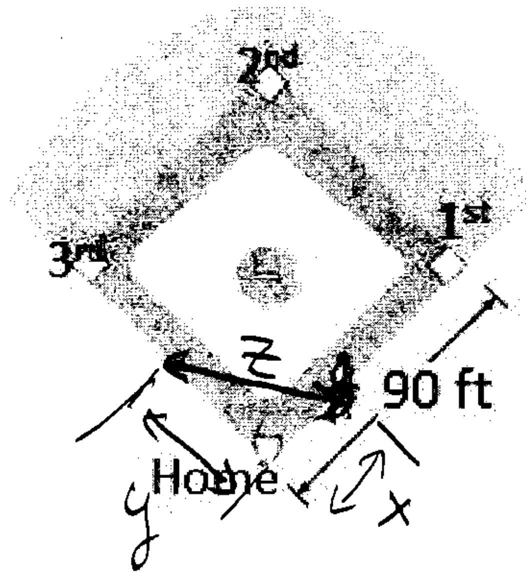
**KNOW**

$\frac{dx}{dt} = 24$ when $x = 30$

$\frac{dy}{dt} = -20$ when $y = 40$

**WANT**

$\frac{dz}{dt} = ?$ at the same instant




$$x^2 + y^2 = z^2$$

$$\Rightarrow \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt}$$

$$x = 30,\ y = 40 \Rightarrow 30^2 + 40^2 = z^2 \Rightarrow z = 50$$

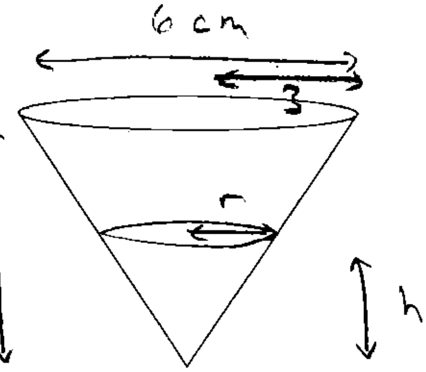
$$\Rightarrow (30)(24) + (40)(-20) = (50)\frac{dz}{dt}$$

$$\Rightarrow 720 - 800 = 50\frac{dz}{dt}$$

$$-80 = 50\frac{dz}{dt}$$

$$\Rightarrow \boxed{\frac{dz}{dt} = -\frac{8}{5} = -1.6 \text{ ft/sec}}$$

6. (10 pts) A conical paper cup is 6 cm across at the top and has a height of 9 cm. Water is pouring into the cup at a rate of 3 $cm^3$/second. How fast is the height of water in the cup rising at the moment when it is 2 cm high? (Recall: the volume of a cone is $V = \frac{1}{3}\pi r^2 h$)




$\boxed{\text{GIVEN}}$ $\frac{dV}{dt} = 3$

$\boxed{\text{WANT}}$ $\frac{dh}{dt} = ?$ WHEN $h = 2$

$\boxed{\text{I}}$ $V = \frac{1}{3}\pi r^2 h$

$\boxed{\text{II}}$ $\frac{r}{3} = \frac{h}{9} \Rightarrow h = 3r$

Thus, $V = \frac{1}{3}\pi r^2 (3r)$

$V = \pi r^3$

$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

$h=2 \Rightarrow 2 = 3r$

$(r = 2/3)$

$\uparrow$

3

$3 = 3\pi \frac{4}{9} \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{9}{4\pi}$

$\Rightarrow \frac{dh}{dt} = 3\frac{dr}{dt} \Rightarrow \boxed{\frac{dh}{dt} = \frac{27}{4\pi} \frac{cm}{sec}}$

$\approx 2.1486 \frac{cm}{sec}$

OR $\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$

AND $\frac{dh}{dt} = 3\frac{dr}{dt}$ PLUG IN