1. (9 pts) Determine the values of the following limits or state that the limit does not exist. If it is correct to say that the limit equals ∞ or $-\infty$, then you should do so. **In all cases, show your work/reasoning. You must use algebraic methods where available. And explain in words your reasoning if an algebraic method is not available.**

(a) $\lim_{x \to 4^+} \frac{5 + \cos(x) + e^x}{\sqrt{x}(4 - x)} = \boxed{-\infty}$

The numerator is approaching $5 + \cos(4) + e^4$ which is positive.
The denominator is approaching 0 through negative values
(becuase $\sqrt{x}(4 - x) < 0$ if $x > 4$)

(b) $\lim_{t \to \infty} \left(3e^{1/t} + \frac{3 + t^2}{5t^2 + \sqrt{1 + 9t^4}} \right)$

$3e^0 = 3$

$$\lim_{t\to\infty} \frac{(3 + t^2)\ \frac{1}{t^2}}{(5t^2 + \sqrt{1+9t^4})\ \frac{1}{t^2}}$$

$$\frac{\sqrt{1+9t^4}}{t^2} = \sqrt{\frac{1+9t^4}{t^4}}$$

$$= \lim_{t\to\infty} \frac{\frac{3}{t^2} + 1}{5 + \sqrt{\frac{1}{t^4} + 9}}$$

$$= \frac{0 + 1}{5 + \sqrt{0 + 9}} = \frac{1}{8}$$

$$3 + \frac{1}{8} = \frac{24}{8} + \frac{1}{8} = \boxed{\frac{25}{8}} = 3.125$$

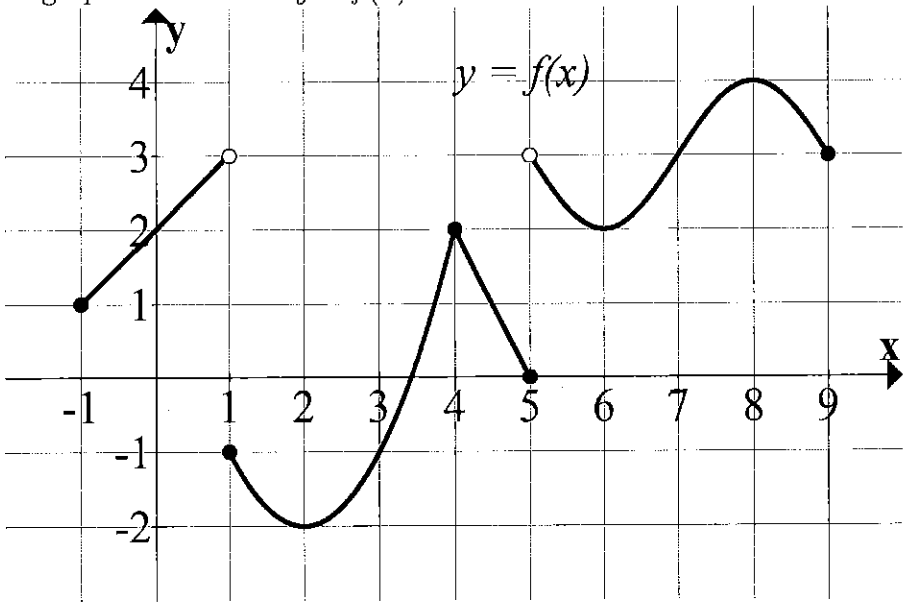
(c) $\lim_{x \to 0} \frac{(\sqrt{1 - x} - 1)}{(\sqrt{4 - x} - 2)} \cdot \frac{\sqrt{1-x} + 1}{\sqrt{1-x} + 1}$

$$= \lim_{x\to 0} \frac{(1 - x - 1)}{(\sqrt{1-x} + 1)(\sqrt{4-x} - 2)} \cdot \frac{(\sqrt{4-x} + 2)}{(\sqrt{4-x} + 2)}$$

$$= \lim_{x\to 0} \frac{-x(\sqrt{4-x} + 2)}{(\sqrt{1-x} + 1)(\cancel{4} - x - \cancel{4})} = \lim_{x\to 0} \frac{\sqrt{4-x} + 2}{\sqrt{1-x} + 1} = \frac{2 + 2}{1 + 1}$$

$$= \boxed{2}$$

2. (12 pts) The graph of a function $y = f(x)$ is shown and is defined for all values $-1 \leq x \leq 9$.



(a) Give all values of x where the derivative, $f'(x)$, is equal to zero.

$\boxed{x = 2, \quad x = 6, \quad x = 8}$

(b) Evaluate the following limits (estimating from the graph and using everything you've learned). If the limit is $\pm\infty$, then say so. If the limit does not exist, then say so.

i. $\lim_{x \to 4} x f(x)$ $= 4 f(4) = 4 \cdot 2 = \boxed{8}$

continuous at $x = 4$

ii. $\lim_{x \to 0} \frac{f(0+h) - f(0)}{h}$ (Hint: You should know what this represents)

$= f'(0) =$ "slope of tangent at $x = 0$" $= \frac{3-1}{1-(-1)} = \frac{2}{2} = \boxed{1}$

iii. $\lim_{x \to 5^-} \frac{x}{f(x)}$ $= \boxed{+\infty}$

The numerator approaches 5. (positive)
The denominator approaches 0, through positive numbers.

3. (10pts)

$\rightarrow 2(x^2+12x+36) = 2x^2+24x+72$

(a) Let $y = 5\tan(x) + 4xe^x + 2(x+6)^2$. Find the equation of the tangent line at $x = 0$.

$y(0) = 5\tan(0) + 4(0)e^0 + 2(0+6)^2 = 72$

$y' = 5\sec^2(x) + 4xe^x + 4e^x + 4x + 24$

$y'(0) = 5\sec^2(0) + 4(0)e^0 + 4e^0 + 4(0) + 24 = 5 + 4 + 24 = 33$

$\boxed{y = 33x + 72}$

(b) Consider the function $g(x) = \begin{cases} \dfrac{(1+x)^2 - 4}{x-1} & \text{, if } x < 1; \\ a\cos(\pi x) + 12\sqrt{x} & \text{, if } x \geq 1. \end{cases}$

Find the value of a that makes the function continuous at $x = 1$.
(Use limits to carefully justify your answer).

$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \frac{(x+1)^2 - 4}{x-1} = \lim_{x \to 1^-} \frac{x^2+2x+1-4}{x-1}$

$= \lim_{x \to 1^-} \frac{(x+3)(x-1)}{(x-1)}$

$= 4$

$\lim_{x \to 1^+} g(x) = g(1) = a\cos(\pi) + 12\sqrt{1} = -a + 12$

WE WANT $-a + 12 \stackrel{?}{=} 4$

$\Rightarrow -a = -8$

$\Rightarrow \boxed{a = 8}$

4. (8 pts) For all parts on this page, let $f(t) = \dfrac{4}{1+5t}$.

(a) Find and completely simplify $\dfrac{f(t+h) - f(t)}{h}$.
(Simplify until the h in the denominator cancels).

$$\frac{\dfrac{4}{1+5(t+h)} - \dfrac{4}{1+5t}}{h} \qquad \frac{(1+5t+5h)(1+5t)}{(1+5t+5h)(1+5t)}$$

$$= \frac{4(1+5t) - 4(1+5t+5h)}{h(1+5t+5h)(1+5t)}$$

$$= \frac{\cancel{4} + 20t - \cancel{4} - \cancel{20t} - 20h}{h(1+5t+5h)(1+5t)}$$

$$= \boxed{\frac{-20}{(1+5t+5h)(1+5t)}}$$

(b) Find the value(s) of t at which the slope of the tangent line to $y = f(t)$ is equal to -5.

Let $h \to 0$ above gives $f'(t) = \dfrac{-20}{(1+5t)(1+5t)} = \dfrac{-20}{(1+5t)^2}$

OR use quotient rule.

WE WANT $\dfrac{-20}{(1+5t)^2} \stackrel{?}{=} -5$

$$\Rightarrow \quad 4 = (1+5t)^2$$

$$\Rightarrow \quad \pm 2 = 1+5t$$

$-2 = 1+5t$ or $2 = 1+5t$

$1 = 5t$

$\Rightarrow -3 = 5t$

$$\boxed{t = -3/5 \quad \text{or} \quad t = 1/5}$$

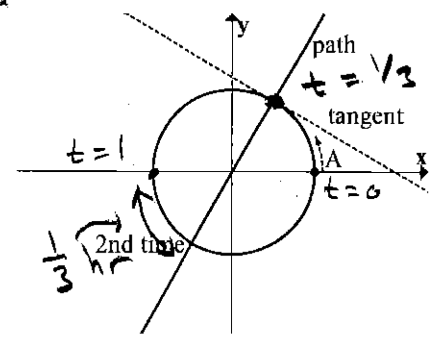
5. (12 pts) $\frac{d}{dt}\left(1 + 2t^{3/2}\right) = 3t^{1/2}$

(a) Let $f(t) = \dfrac{t^2+7}{1+2\sqrt{t^3}}$. Find $f'(1)$.

$$f'(t) = \frac{(1+2\sqrt{t^3})(2t) - (t^2+7)(3t^{1/2})}{(1+2\sqrt{t^3})^2}$$

$$f'(1) = \frac{(1+2)(2) - (8)(3)}{(3)^2} = \frac{-6}{3} = \boxed{-2}$$

(b) . Andy is jogging around a circular loop with radius 4 miles ($x^2 + y^2 = 16$). His location after t hours is given by $x = 4\cos(\pi t)$, $y = 4\sin(\pi t)$. A straight line path runs across the loop and is given by the equation $y = \sqrt{3}x$. (all shown at right)



i. Find the equation for the tangent line to the circle the first time Andy crosses the path (this tangent line is shown in the picture).

TANGENT SLOPE $= -\frac{1}{\sqrt{3}}$

NOW WE NEED THE POINT OF INTERSECTION OF $y = \sqrt{3}x$ AND $x^2+y^2 = 16 \Rightarrow x^2 + 3x^2 = 16$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$y = \sqrt{3}\cdot 2 = 2\sqrt{3}$

slope $= -\frac{1}{\sqrt{3}}$, POINT $= (2, 2\sqrt{3})$

LINE: $\boxed{y = -\frac{1}{\sqrt{3}}(x-2) + 2\sqrt{3}}$

ii. Find the second time, t, where Andy crosses the path.

$x = 4\cos(\pi t)$, $y = 4\sin(\pi t)$ $\Rightarrow$ ONE LOOP IN 2 HOURS
HALF LOOP IN 1 HOUR

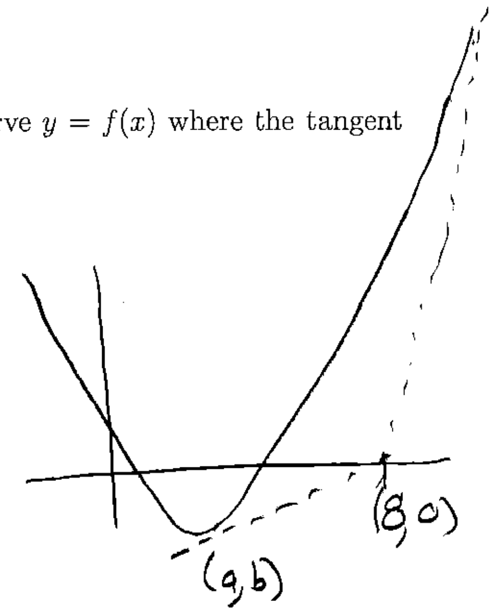
FIRST TIME: $4\cos(\pi t) = 2 \Rightarrow \cos(\pi t) = 1/2 \Rightarrow \pi t = \pi/3$

$\Rightarrow t = 1/3$ HOUR

BY SYMMETRY, SECOND TIME IS

$\boxed{t = 1 + \frac{1}{3} = 4/3 \text{ HOUR}}$

6. (8 pts) Let $f(x) = x^2 - 5x + 1$. There are two points on the curve $y = f(x)$ where the tangent line at that point would also have an x-intercept of 8.
Find the coordinates $(x, y) = (a, b)$ of the two points of tangency.



$\boxed{\text{I}}$ (a,b) IS ON THE CURVE

$\Rightarrow \quad b = a^2 - 5a + 1$

$\boxed{\text{II}}$ DESIRED SLOPE $= \dfrac{b-0}{a-8}$

$\boxed{\text{III}}$ $f'(x) = 2x - 5$

TANGENT SLOPE AT $x = a$ $= 2a - 5$

WE WANT

(i) $b = a^2 - 5a + 1$

(ii) $2a - 5 = \dfrac{b}{a-8} \Rightarrow (2a-5)(a-8) = b$

THUS,

$$2a^2 - 5a - 16a + 40 \stackrel{?}{=} a^2 - 5a + 1$$

$$\Rightarrow \quad a^2 - 16a + 39 \stackrel{?}{=} 0$$

$$\Rightarrow \quad (a - 13)(a - 3) = 0$$

$\Rightarrow \quad a = 3$ OR $a = 13$

$\Downarrow$

$b = (3)^2 - 5(3) + 1$

$b = -5$

$\Downarrow$

$b = (13)^2 - 5(13) + 1$

$b = 169 - 65 + 1 = 105$

$$\boxed{(a,b) = (3,-5) \quad \text{OR} \quad (13,105)}$$