1. (13 pts) Put a box around your final answer. You do not have to simplify.

(a) Find y' for $y = \left(\ln(t^4+1)\right)^8$

$$\boxed{y' = 8\left(\ln(t^4+1)\right)^7 \frac{1}{t^4+1} \cdot 4t^3}$$

(b) Find $f'(x)$ for $f(x) = \frac{1}{4} + 7x + \frac{3}{4e^{\sqrt{x}}}$ $= \frac{1}{4} + 7x + \frac{3}{4}e^{-x^{1/2}}$

$$\boxed{f'(x) = 7 + \frac{3}{4}e^{-x^{1/2}} \cdot \left(-\tfrac{1}{2}x^{-1/2}\right)}$$

(c) Find the general anti-derivative: $\int \frac{\sqrt{x}}{7} - 4e^{3x}\, dx$

$$= \int \frac{1}{7}x^{1/2} - 4e^{3x}\, dx$$

$$= \boxed{\frac{1}{7}\cdot\frac{2}{3}x^{3/2} - \frac{4}{3}e^{3x} + C}$$

(d) Evaluate $\int_1^2 x\left(\frac{8}{x^3} + \frac{2}{x}\right) dx$ $= \int_1^2 8x^{-2} + 2\, dx$

$$= \frac{8}{-1}x^{-1} + 2x \Big|_1^2$$

$$= -\frac{8}{x} + 2x \Big|_1^2$$

$$= \left(-\frac{8}{(2)} + 2(2)\right) - \left(-\frac{8}{(1)} + 2(1)\right)$$

$$= (-4+4) - (-8+2)$$

$$= \boxed{6}$$

2. (12 pts) Two balloons are at the same height at $t = 0$. Time, t, is measured in minutes and height is measured in feet. You are given:

$$A'(t) = 12 - \frac{3t}{2} \qquad \text{feet/min} \quad = \text{'RATE of ascent for balloon A'}$$

$$B(t) = \tfrac{1}{3}t^3 - 4t^2 + 15t + 40 \quad \text{feet} \qquad = \text{'HEIGHT for balloon B'}$$

(a) Use the fact that $A(0) = B(0)$ to find the formula for $A(t)$ without any undetermined constants.

$A(t) = \int 12 - \frac{3}{2}t \, dt = 12t - \frac{3}{4}t^2 + C$

$A(0) = B(0) = 40$

$\Rightarrow 12(0) - \frac{3}{4}(0)^2 + C = 40 \Rightarrow C = 40$

ANSWER: $A(t) = \underline{12t - \frac{3}{4}t^2 + 40}$

(b) Give an interval over which the graph of the height of Balloon B is concave down.

$B'(t) = t^2 - 8t + 15$

$B''(t) = 2t - 8 \overset{?}{=} 0 \Rightarrow t = 4$

B: $\cap$ | $\cup$

B'': $-$ 4 $+$

$t = \underline{\quad 0 \quad}$ to $t = \underline{\quad 4 \quad}$

(c) Find all times at which Balloon B changes from falling to rising.

$B'(t) = t^2 - 8t + 15 = (t-3)(t-5) = 0$

$t = 3$ or $t = 5$

B: $\nearrow$ | $\searrow$ | $\nearrow$

B': $+$ 3 $-$ 5 $+$

$t = \underline{\quad 5 \quad}$ min

(d) Find the lowest and highest altitudes reached by Balloon A from $t = 0$ to $t = 10$.

$A'(t) = 12 - \frac{3}{2}t \overset{?}{=} 0 \Rightarrow 24 - 3t = 0 \Rightarrow t = 8$

$A(0) = 40$

$A(8) = 12(8) - \frac{3}{4}(8)^2 + 40 = 88$

$A(10) = 12(10) - \frac{3}{4}(10)^2 + 40 = 85$

ANSWER: 'lowest altitude' = $\underline{\quad 40 \quad}$ feet

'highest altitude' = $\underline{\quad 88 \quad}$ feet

3. (12 pts) You sell items. The functions for marginal revenue and marginal cost (in dollars/item) are given by

$$MR(q) = 9e^{0.02q} \text{ and } MC(q) = q^2 - 16q + 126,$$

where q is in thousands of items. You are also told that Fixed Costs are given $FC = 15$ thousand dollars (so $TC(0) = 15$).

(a) Give the functions for Total Revenue and Total Cost (solve for the constants of integration).

$TR(q) = \int 9e^{0.02q}\,dq = \frac{9}{0.02} e^{0.02q} + C = 450e^{0.02q} + C$

$TR(0) = 0 \Rightarrow 450 + C = 0 \Rightarrow C = -450$

$TC(q) = \int q^2 - 16q + 126\,dq = \frac{1}{3}q^3 - 8q^2 + 126q + C$

$TC(0) = 15 \Rightarrow C = 15$

ANSWER: $TR(q) =$ $450e^{0.02q} - 450$

$TC(q) =$ $\frac{1}{3}q^3 - 8q^2 + 126q + 15$

(b) Find the largest and smallest values of Marginal Cost on the interval $q = 0$ to $q = 10$.

$MC'(q) = 2q - 16 \stackrel{?}{=} 0 \Rightarrow q = 8$

$MC(0) = 126$

$MC(8) = (8)^2 - 16(8) + 126 = 62$

$MC(10) = (10)^2 - 16(10) + 126 = 66$

ANSWER: 'smallest value of MC' = 62 dollars/item

'largest values of MC' = 126 dollars/item

(c) Recall: $AC(q) = \frac{TC(q)}{q}$.
Determine if $AC(q)$ is concave up, concave down, or neither at $q = 1$ thousand items. (You must show appropriate derivatives and make correct conclusions to get full credit).

$AC(q) = \frac{1}{3}q^2 - 8q + 126 + 15q^{-1}$

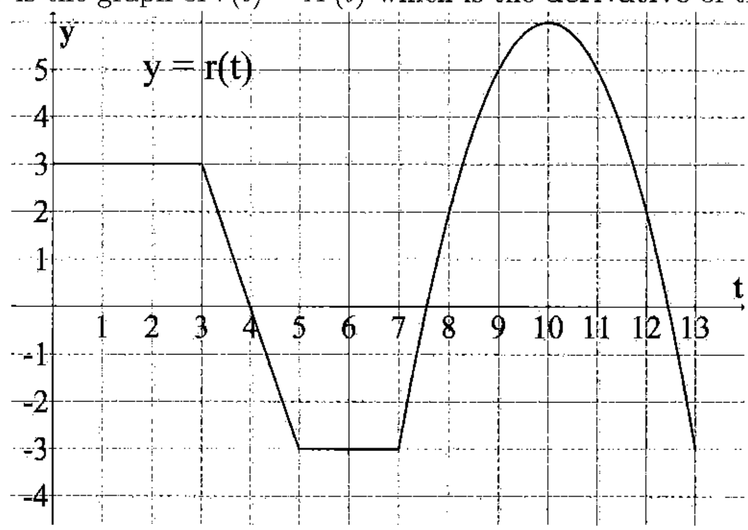
$AC'(q) = \frac{2}{3}q - 8 - 15q^{-2}$

$AC''(q) = \frac{2}{3} + 30q^{-3}$

$AC''(1) = \frac{2}{3} + 30 > 0$

ANSWER: (Circle one) CONCAVE UP CONCAVE DOWN NEITHER

4. (13 pts) The graph below shows the **rate of ascent**, $r(t)$, at time t for a hot-air balloon. Let $A(t)$ be the function for the height (in feet) of the hot-air balloon at time t minutes. As a reminder, the picture below is the graph of $r(t) = A'(t)$ which is the **derivative** of the altitude function!!



Use the picture to estimate the answers to the questions below as accurately as possible.

(a) Estimate the following:

i. $\int_0^4 r(t)\,dt = 9 + \frac{1}{2}(1)(3) = 9 + 1.5 = \boxed{10.5}$

ii. $\int_3^7 r(t)\,dt = 0 + 2(-3) = \boxed{-6}$

$(3,3)$, $(4,0)$ $\frac{3-0}{3-4} = -3$

iii. $A''(4) = r'(4) =$ slope at 4 $= \boxed{-3}$

(b) Give the longest interval of time over which the graph of $A(t)$ is concave up (*remember* the picture above is $A'(t)$). WANT $A''(t)$ POSITIVE AND $A''(t) = r'(t) =$ slope.

WANT POSITIVE SLOPE!

ANSWER: $t =$ 7 min to $t =$ 10 min

(c) Find *all* critical values of $A(t)$ (estimate from the picture).

ANSWER: $t =$ 4, 7.2, 12.2 min

(d) At time $t = 0$, assume the balloon is 30.5 feet high. Give the time and the corresponding altitude at which the balloon is highest in the first 7 minutes.

A ↗ ↘

A' + 4 −

↑ MAX AT 4

$\underbrace{\int_0^4 r(t)\,dt}_{10.5} = A(4) - A(0)$

$10.5 = A(4) - 30.5 \Rightarrow A(4) = 41$

ANSWER: $t =$ 4 min

max height = 41 feet