9.4 Overview

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections.

9.4: Instantaneous Rates of Change and Tangent Lines

1. *Notational Note*: There are several common notations for the derivative. It is good to be aware of all of them. If $y = f(x)$, then here are the four most common derivative notations you will see (all mean the same thing).

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}[f(x)] = \text{“the derivative of } f(x)\text{.”}$$

2. Using the techniques from 9.3 (*i.e.* simplifying $\frac{f(x+h)-f(x)}{h}$), we can find slope of tangent (derivative) formulas. For example you should be able to quickly verify that:

   - If $f(x) = x^2$, then $f'(x) = 2x$.
   - If $f(x) = x^3$, then $f'(x) = 3x^2$ (I do this one as the last example in the 9.3 review sheet).
   - If $f(x) = x^4$, then $f'(x) = 4x^3$ (This takes a bit more time to verify because of the algebra in expanding, but it all comes from expanding $\frac{(x+h)^4-x^4}{h}$).
   - If $f(x) = x^5$, then $f'(x) = 5x^4$.

   There is a pattern here:

   $$\boxed{\text{POWER RULE: If } f(x) = x^n\text{, then } f'(x) = nx^{n-1}.}$$

   Some quick examples:

   - If $y = x^{3.5}$, then $y' = 3.5x^{2.5}$.
   - If $y = \sqrt{x}$ (which is the same as $x^{1/2}$), then $y' = \frac{1}{2}x^{-1/2}$.
   - If $y = \frac{1}{x^4}$ (which is the same as x^{-4}), then $y' = -4x^{-5}$.

   We have seen this pattern for positive whole numbers, but it actually works for ALL numbers n (integers, fractions, decimals, whatever). So any function that can be simplified and written in the form x^{BLAH} where BLAH is a number has a derivative formula that looks like $BLAHx^{BLAH-1}$. A couple special cases are worth note.

   If the function is just a constant horizontal line (for example, $f(x) = 4$), then the slope is zero everywhere on the line giving:

   $$\boxed{\text{CONSTANT FUNCTION RULE: If } f(x) = c\text{, then } f'(x) = 0.}$$

   If the function is the line $f(x) = x$ (which is the same as $f(x) = x^1$), the rule still holds because the slope is 1 everywhere and $x^0 = 1$, so you get:

   $$\boxed{\text{NOTABLE SPECIAL CASE: If } f(x) = x\text{, then } f'(x) = 1.}$$

3. Another important observation is that if a coefficient is in front of a function, then the slope is just multiplied by that coefficient. Namely:

   $\boxed{\text{COEFFICIENT RULE: The derivative of } cf(x) \text{ is } cf'(x). \text{ (``Coefficients come alone for the ride.'')}}$

   You can see this in the example $f(x) = 10x^2$ because if you started to find the derivative you should get $\frac{f(x+h)-f(x)}{h} = \frac{10(x+h)^2-10x^2}{h} = 10\frac{(x+h)^2-x^2}{h}$ (I factored out the 10). So the derivative of $10x^2$ will be exacly 10 times whatever the derivative of x^2. Since we know the derivative of x^2 is $2x$, we end up with $10 \cdot 2x = 20x$.
   Some quick examples:

   - If $y = 10x^2$, then $y' = 10 \cdot 2x = 20x$.
   - If $y = 4x^{5/4}$, then $y' = 4\frac{5}{4}x^{1/4} = 5x^{1/4}$.
   - If $y = \frac{7}{x^2}$ (which is the same as $7x^{-2}$), then $y' = 7 \cdot (-2)x^{-3} = -14x^{-3}$.

4. The last of the basic fundamental rules involves sums and differences. If slope of the tangent for the sum of two functions is the sum of the their slopes (also true for differences). Namely,

   $\boxed{\text{SUM RULE: The derivative of } f(x) + g(x) \text{ is } f'(x) + g'(x).}$

   $\boxed{\text{DIFFERENCE RULE: The derivative of } f(x) - g(x) \text{ is } f'(x) - g'(x).}$

   You can see this in the example $f(x) = x^2 + x^3$ because if you started to find the derivative you should get $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+(x+h)^3-x^2-x^3}{h} = \frac{(x+h)^2-x^2}{h} + \frac{(x+h)^3-x^3}{h}$. So the derivative of $x^2 + x^3$ will be the sum of the derivatives of x^2 and x^3. Since we know those derivative we get $2x + 3x^2$.
   Some quick examples:

   - If $y = x^2 + x^3$, then $y' = 2x + 3x^2$.
   - If $y = x^2(5 - x^2)$ (which is the same as $5x^2 - x^4$), then $y' = 10x - 4x^3$.
   - If $y = \frac{1}{x} + x^{14}$ (which is the same as $x^{-1} + x^{14}$), then $y' = -x^{-2} + 14x^{13}$.

5. In summary, here is how you use these rules:

   (a) Expand and try to seperate each term by sums or differences if you can.

   (b) Rewrite each terms at cx^b.

   (c) The coefficients come along for the rule.

   (d) Use the power rule.

   Example: $f(x) = x(x^3 - 4x) + \frac{5}{x} - 7\sqrt[3]{x^2}$

   (a) Expand: $x^4 - 4x^2 + \frac{5}{x} - 7\sqrt[3]{x^2}$

   (b) Rewrite: $x^4 - 4x^2 + 5x^{-1} - 7x^{2/3}$

   (c) Coefficients: The answer will look like $??? - 4??? + 5??? - 7???$

   (d) Power Rule: $4x^3 - 4 \cdot 2x + 5 \cdot (-1)x^{-2} - 7 \cdot \frac{2}{3}x^{-1/3}$

   If we are going to analyze our derivative, then typically we simplify our final answer to get $f'(x) = 4x^3 - 8x - 5x^{-2} - \frac{14}{3}x^{-1/3}$.