9.3 Overview

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections.

9.3: Instantaneous Rates of Change and Tangent Lines

1. Recall that
$$\frac{f(b)-f(a)}{b-a} = \text{“average rate of change from } x=a \text{ to } x=b\text{”}$$
If the function is distance, then the rate is average speed.
If the function is revenue and the change in quantity is one unit, then the rate is marginal revenue.
If the function is cost and the change in quantity is one unit, then the rate is marginal cost.
This rate can be viewed graphically as the slope of the secant line that goes through the graph at $x = a$ and $x = b$.

2. In this course, we are interested in the instantaneous rate at a point on a graph. Graphically, we are looking for the slope of the tangent line to $f(x)$ at a particular point. We use the notation:
$$f'(x) = \text{“the slope of the tangent line to } f(x) \text{ at } x\text{”}.$$
If the function $D(x)$ is distance, then $D'(x)$ is instantaneous speed (or speedometer speed).
If the function $R(x)$ is revenue and one unit is small on the graph, then $R'(x)$ is very close to marginal revenue (often this is taken as the definition of marginal revenue in business classes).
If the function $C(x)$ is cost and one unit is small on the graph, then $C'(x)$ is very close to marginal cost (often this is taken as the definition of marginal cost in business classes).

3. We can't use the average rate formula because the denominator would be zero, but we can use average rate between the value and a value "nearby" to estimate the instantaneous rate.
For example, if we wanted to estimate the tangent slope at $x = 8$, we could compute
$$f'(8) \approx \frac{f(8.001)-f(8)}{0.001}$$
If that is not accurate enough, then we could pick a number even closer to 8, like 8.000000001.

4. We can do even better and make our life simpler if we are systematic. Instead of adding 0.1 or 0.0001 or 0.00000001, we could just replace this small amount with a variable, we like to use h, and simplify BEFORE we plug in. This requires some algebra up front, but dramatically simplifies our work later.

   So we note that if h is small, then
$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$
The closer and closer that h gets to zero, the more accurate this approximation gets.

5. For a given function $f(x)$, if we can find $\frac{f(x+h)-f(x)}{h}$ and simplify it as much as possible (hopefully to a place where the h in the denominator cancels), then we can find an EXACT value for the instantaneous rate by letting $h \to 0$. This will give us the formula for the derivative function, which will be the formula that will give that instantaneous rate of change (*e.g.* tangent slope) at any value of x.

Finding Tangent Slope Examples:
In the examples below, we find the formula for the slope of the tangent by first finding the formula for the slope of the secant from x to $x+h$, then letting $h \to 0$. This process can be used to find the exact formula for the slope of tangent at any point for ANY function. In 9.4, 9.5 and 9.6, we will learn some shortcuts for certain cases so that we can be faster, but, in all cases, the formula for the slope of the tangent can always be found this way.

1. If $f(x) = 3x + 1$, then we get the instantaneous rate formula as follows:
   (a) Write $\frac{f(x+h)-f(x)}{h} = \frac{[3(x+h)+1]-[3x+1]}{h}$
   (b) Simplify until h in the denominator is gone,

$$\begin{aligned}
\frac{[3(x+h)+1]-[3x+1]}{h} &= \frac{[3x+3h+1]-[3x+1]}{h} && \text{distribute the 3}\\
&= \frac{3x+3h+1-3x-1}{h} && \text{distribute the negative}\\
&= \frac{3h}{h} && \text{cancel in the numerator}\\
&= 3 && \text{cancel } h
\end{aligned}$$

   (c) We conclude that $f'(x) = 3$, *e.g.* the slope of $f(x) = 3x + 1$ is always a constant 3 at all points (we should have already known that since $f(x)$ is just a line).

2. If $f(x) = 4x^2 - 3$, then we get the instantaneous rate formula as follows:
   (a) Write $\frac{f(x+h)-f(x)}{h} = \frac{[4(x+h)^2-3]-[4x^2-3]}{h}$
   (b) Simplify until h in the denominator is gone,

$$\begin{aligned}
\frac{[4(x+h)^2-3]-[4x^2-3]}{h} &= \frac{[4(x^2+2xh+h^2)-3]-[4x^2-3]}{h} && \text{expand the square}\\
&= \frac{[4x^2+8xh+4h^2-3]-[4x^2-3]}{h} && \text{distribute the 4}\\
&= \frac{4x^2+8xh+4h^2-3-4x^2+3]}{h} && \text{distribute the negative}\\
&= \frac{8xh+4h^2}{h} && \text{cancel in the numerator}\\
&= \frac{8xh}{h} + \frac{4h^2}{h} && \text{distribute}\\
&= 8x + 4h && \text{cancel}
\end{aligned}$$

   (c) If we let $h \to 0$, we conclude that $f'(x) = 8x$, *e.g.* the slope of $f(x) = 4x^2 - 3$ is always given by $8x$ at all points.

3. If $f(x) = x^2 - 5x + 2$, then we get the instantaneous rate formula as follows:
   (a) Write $\frac{f(x+h)-f(x)}{h} = \frac{[(x+h)^2-5(x+h)+2]-[x^2-5x+2]}{h}$
   (b) Simplify until h in the denominator is gone,

$$\begin{aligned}
\frac{[(x+h)^2-5(x+h)+2]-[x^2-5x+2]}{h} &= \frac{[x^2+2xh+h^2-5(x+h)+2]-[x^2-5x+2]}{h} && \text{expand the square}\\
&= \frac{[x^2+2xh+h^2-5x-5h+2]-[x^2-5x+2]}{h} && \text{distribute the 5}\\
&= \frac{x^2+2xh+h^2-5x-5h+2-x^2+5x-2}{h} && \text{distribute the negative}\\
&= \frac{2xh+h^2-5h}{h} && \text{cancel in the numerator}\\
&= \frac{2xh}{h} + \frac{h^2}{h} - \frac{5h}{h} && \text{distribute}\\
&= 2x + h - 5 && \text{cancel}
\end{aligned}$$

   (c) If we let $h \to 0$, we conclude that $f'(x) = 2x - 5$, *e.g.* the slope of $f(x) = x^2 - 5x + 2$ is always given by $2x - 5$ at all points.

4. If $f(x) = x^3$, then we get the instantaneous rate formula as follows:

(a) Write $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3-x^3}{h}$

(b) Simplify until h in the denominator is gone,

$$\begin{aligned}
\frac{(x+h)^3-x^3}{h} &= \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} && \text{expanding } (x+h)^3\text{, do this on a separate sheet} \\
&= \frac{3x^2h+3xh^2+h^3}{h} && \text{cancel in the numerator} \\
&= \frac{3x^2h}{h} + \frac{3xh^2}{h} + \frac{h^3}{h} && \text{distribute} \\
&= 3x^2 + 3xh + h^2 && \text{cancel}
\end{aligned}$$

(c) If we let $h \to 0$, we conclude that $f'(x) = 3x^2$, *e.g.* the slope of $f(x) = x^3$ is always given by $3x^2$ at all points.