13.4 Overview

In this section we discuss more applications of integrals specific to business. Namely, income flow, consumer surplus and supplier surplus.

Income Flow: If you know a formula for the *rate*, $r(t)$, at which income in coming in, then you can find the total amount of income by integrating

$$\text{Total income from 0 to } k \ = \int_0^k r(t)\,dt$$

where k needs to be in the same units as t. For example: if t is in years, then k needs to be the number of years from the beginning. This is a direct application of the fundamental theorem of calculus which says that the definite integral of a derivative gives the change in any antiderivative, but it is a situation where you might see calculus pop up in business.

Here are a few simple case study type examples:

1. *Constant Derivative*: Assume you are paid \$3,000 per month and that amount never changes. How much money will you make in one years?

   ANSWER: You don't need calculus for this problem since it is a constant rate. You could just do \$3,000 times 12 months which is \$36,000. But I wanted to start with a simple example so that you could see that the integral gives the same thing.

   The derivative is a constant $r(t) = 3000$, so to get the total in 12 months we compute

   $$\text{Total income from 0 to 12 months} \ = \int_0^{12} 3000\,dt = 3000t|_0^{12} = 3000(12) - 3000(0) = \$36,000.$$

   You can see the integral gives the same answer.

   Aside: Note when the derivative is constant, the amount grows linearly. The formula for the amount would be $I(t) = 3000t$.

2. *Linear Derivative*: Your companies profits have been growing. In the first four years, your profits were about \$5,000, \$15,000, \$30,000, and \$50,000 respectively. Here were the increases per year from year to year: \$5,000, then \$10,000, then \$15,000, then \$20,000. In other words, the *rate* of growth is increasing by a constant amount, namely \$5,000 per year each year.

   Written as a formula we have the derivative is roughly given by $r(t) = 5000t$ dollars per year.

   How much money in profit will we make in the first 10 years?

   ANSWER:

   $$\text{Total income from 0 to 10 years} \ = \int_0^{10} 5000t\,dt = 2500t^2\Big|_0^{10} = 2500(10)^2 - 2500(0)^2 = \$250,000.$$

   Aside: Note that linear derivative means the income grows quadratically, meaning $I(t) = 2500t^2$.

3. *Exponential Derivative*: Money that is compounding grows exponentially (the derivative and the total amount both grow exponentially). You should already know that if a principal amount, P is deposited into a bank account at a decimal interest rate of r compounded annually, then the amount in t years is given by $A(t) = Pe^{rt}$. So it's derivative is $A'(t) = rPe^{rt}$.

   Assume you have deposited $1,000 in principal into an account that makes 4.5% annually, compounded continuously. So the amount and derivatives formulas are $A(t) = 1000e^{0.045t}$ and $A'(t) = 45e^{0.045t}$. How much income (interest) will you get from this account in the first 5 years?

   ANSWER:

$$\text{Total income (interest) from 0 to 5 years } = \int_0^5 45e^{0.045t}\,dt$$

$$= 1000e^{0.045t}\Big|_0^5 = 1000e^{0.045(5)} - 1000e^{0.045(0)} = \$1,252.32 - \$1,000 = \$252.32.$$

   So we see that integral calculus gives us a general way to get total amounts from rates which is useful in general and helpful here specifically for income flows.

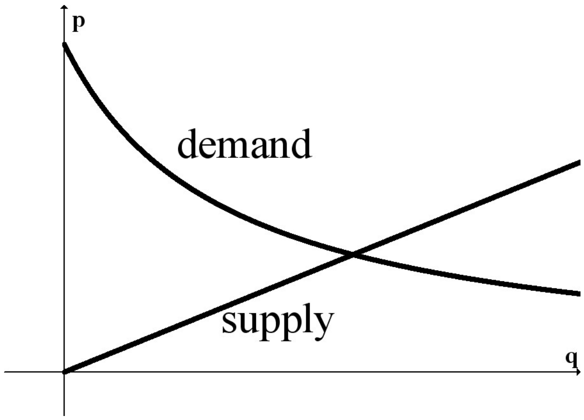
Consumer and Supplier Surplus

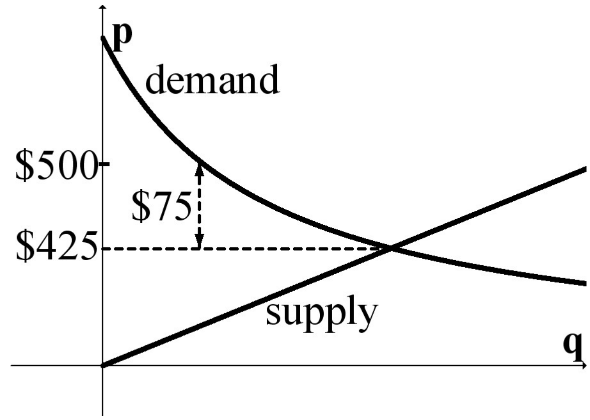
First, some reminders about supply and demand

- The **demand function** for a particular product gives a relationship between the price per unit, p, and the quantity, q, that consumers will purchase at that price. One basic property of a demand function is that when price goes up, demand goes down (and when demand goes up, price goes down). So it's a decreasing function.
- The **supply function** for a particular product gives a relationship between the price per unit, p, and the quantity, q, that the manufacturers are willing to sell at that price. One basic property of a supply function is that when the price goes up, the manufacturers are willing to produce and sell more. So it's an increasing function.
- The price and quantity at which the supply and demand curves intersect is called the **market equilibrium**. That is the price and quantity and which the manufacturer's are willing to produce and sell exactly the amount that consumers are willing to purchase at that price. (under 'pure competition' this should be the price of the product).

The **consumer surplus** concept:
Let's say you go to the store to buy some new piece of technology. You go ready to spend $500, but when you get to the store you find that it is on sale for $425. From your perspective, you happily saved $75, from the store's perspective, they could have made $75 more from you if they would have known how desperate you were for that new piece of technology. In this little scenario, we say that you have a personal consumer surplus of $75.

Taking this example further, if $425 is the market equilibrium price, then your willingness to buy at $500 means you are part of the demand curve that comes before the equilibrium (your one of the people that was willing to pay more). And you can visualize this $75 amount at the distance between the demand curve and the horizontal equilibrium line at $425. Here is a picture supply and demand, then a picture of the market equilibrium line and the surplus of $75 that I am talking about in this example:

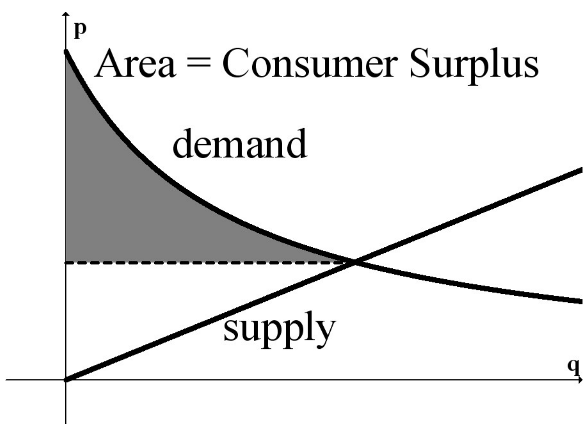




The total amount of money that could be obtained by selling over the market price to all the people that would be willing to pay more than market equilibrium is called the **consumer surplus**. We can find this by adding up all the surpluses using calculus. That is, the consumer surplus is the area below the demand curve and above the horizontal market equilibrium price line. Thus

Consumer Surplus = (Area under demand from 0 to equilibrium) - (Area of rectangle under equilibrium line)

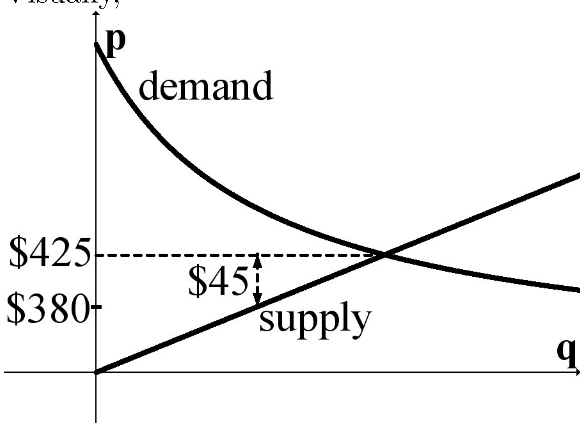
Visually,

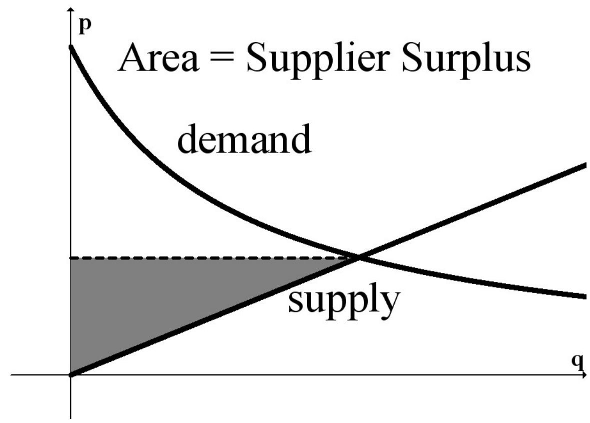


The **supplier surplus** (or **producer surplus**) concept:
A very similar analysis can be done for the supplier. If a supplier produces and sells a produce for less than market equilibrium, then they are happy when they find out that they can sell for more (or unhappy if they sell for the price they intended and realize they were giving too much of a bargain).
Assume a different supplier produced and sold the same bit of technology from the last example. They had planned to sell it for \$380 at a different store and as a result had produced fewer quantities than most other stores did. Then they found out that the market equilibrium price is \$425, so they didn't produce enough to meet that equilibrium, but they can sell for a \$45 surplus from what they had originally planned. We call this a suppliers surplus.
The total amount of money in seller surpluses is called the **supplier surplus**. That is, the supplier surplus is the area above the supply curve and below the horizontal market equilibrium price line. Thus

Supplier Surplus = (Area of rectangle under equilbium line) - (Area under supply from 0 to equilibrium)

Visually,





Example: Assume the demand function is $p = 196 - x^2$ and the supply function is $p = x^2 + 4x + 126$. Find the consumer and producer surplus.

ANSWER:
First, we find market equilibrium. Supply and demand are equal when $196 - x^2 = x^2 + 4x + 126$ giving $0 = 2x^2 + 4x - 70 = x^2 + 2x - 35$ which has solutions $x = -7$ and $x = 5$. So

$$\text{market equilibrium: } x = 5 \text{ and } p = 196 - (5)^2 = 171.$$

Next, note that the area of the rectangle formed by the horizontal market equilibrium line is $5 \cdot 171 = 855$. And we are ready to compute:

$$\text{Consumer Surplus } = \int_0^5 196 - x^2\,dx - 855 \text{ and Producer Surplus } = 855 - \int_0^5 x^2 + 4x + 126\,dx$$

You can compute these to verify that the numbers you get are as below:

$$\text{Consumer Surplus } = 938.33 - 855 = \$83.33 \text{ and Producer Surplus } = 855 - 721.67 = \$133.33.$$