

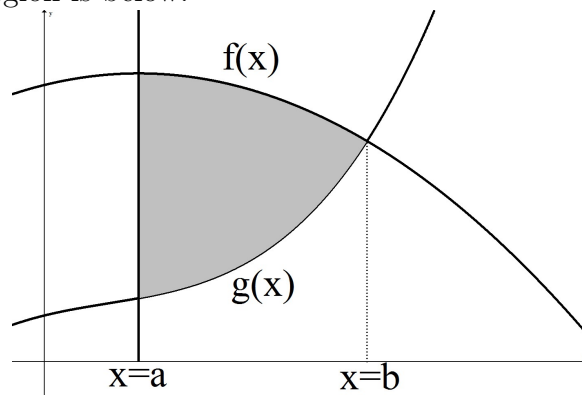
13.3 Overview

In this section we learn how to find the area between curves, which can be used to find profit from MR/MC and to find distance between two cars/balloons/amounts when given both their derivatives. You should be able to use these facts given graphs or given formulas.

To find the area of a region that satisfies:

- $y = f(x)$ is the formula for the curve that is always on top.
- $y = g(x)$ is the formula for the curve that is always on bottom.
- $x = a$ is the farthest x -value on the left.
- $x = b$ is the farthest x -value on the right.

A visual example of such a region is below:



The **main fact** of the section is that these areas can be found by using

$$\text{Area between curves} = \int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b f(x) - g(x) dx$$

Applications:

- *MR and MC*: When given $MR(x)$ and $MC(x)$. We know the following (the first three facts are review, the new fact is number 4):

1. $TR(x)$ is the area under $MR(q)$ from 0 to x . In formula format, we are saying

$$TR(x) = \int_0^x MR(q) dq.$$

You don't need to be given an initial value of TR because it is always the case that $TR(0) = 0$ (and the accumulated area starts at zero as well).

2. $VC(x)$ is the area under $MC(q)$ from 0 to x . In formula format, we are saying

$$VC(x) = \int_0^x MC(q) dq.$$

Again, you don't need to be given an initial value of VC because $VC(0) = 0$.

3. To find $TC(x)$ you must be given some initial condition so that you can find the fixed costs, FC . Since $TC(x) = VC(x) + FC$, we can say that

$$TC(x) = \int_0^x MC(q) dq + FC.$$

4. **NEW**: Finally, since profit satisfies $P(x) = TR(x) - TC(x)$, we can say that

$$P(x) = \int_0^x MR(q) dq - \int_0^x MC(q) dq - FC = \int_0^x MR(q) - MC(q) dq - FC$$

In words, this means:

Profit = 'the signed area between MR and MC from 0 to x ' - Fixed Costs.

When I say this, I mean

- If $MR > MC$, then we treat the area between as positive (it is positive profit).
- If $MR < MC$, then we treat the area between as negative (it is negative profit).

Moral: If you want to visualize profit when given MR and MC graphs, then look at areas between them.

- *Two Speed Graphs*: Suppose you are given the speed graphs $A'(t)$ and $B'(t)$ for two cars (or the rate of ascent for two balloons or the rate of flow for two vats or the rate of change for some other amount). We know that:

“Change in distance from $t = c$ to $t = d$ for car A ” = “Area under $A'(t)$ from c to d ”

“Change in distance from $t = c$ to $t = d$ for car B ” = “Area under $B'(t)$ from c to d ”

In equation format, we say:

$$A(d) - A(c) = \int_c^d A'(t) dt$$

$$B(d) - B(c) = \int_c^d B'(t) dt$$

NEW:

“Change in distance from $t = c$ to $t = d$ ” = “Signed area between $A'(t)$ and $B'(t)$ from c to d ”

In equation format, let $F(t) = A(t) - B(t)$. Writing it this way means $F(t)$ is the distance that A is ahead of B which can be positive or negative depending on if A is ahead or behind. Then

$$\text{“Change in distance”} = F(d) - F(c) = \int_c^d A'(t) - B'(t) dt$$

Note that I have said this in the form $A(t) - B(t)$ instead of $B(t) - A(t)$, you don't have to do it the way I did it. But if you are doing $A(t) - B(t)$, then you need to treat $A'(t)$ as the ‘top’ function and $B'(t)$ as the ‘bottom’ function meaning:

- If $A' > B'$, then we treat the area between as positive (A is going faster than B , so A is either getting farther ahead or A is catching up to B . Either way $F(t) = A(t) - B(t)$ is going up.).
- If $A' < B'$, then we treat the area between as negative (A is going slower than B , so A is either getting farther behind or B is catching up to A . Either way $F(t) = A(t) - B(t)$ is going down.).

Moral: If you want to visualize differences **between amounts** of A and B when given A' and B' graphs, then look at areas between them.

Examples:

1. Find the area of the shaded region shown at the right.

Answer:

First we need to find where they intersect (so we know the x values on the left and right).

$$3x = x^2 \Rightarrow 0 = x^2 - 3x = x(x - 3)$$

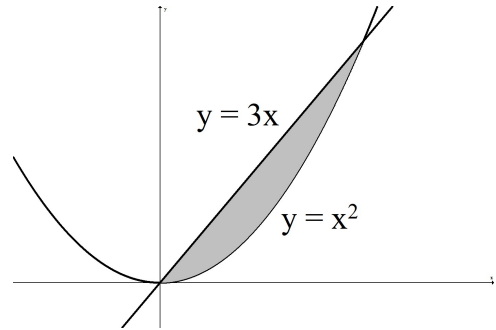
so $x = 0$ and $x = 3$ are the intersection points.

Now we set up our integral:

$$\text{Area} = \int_0^3 3x - x^2 dx$$

Evaluating gives

$$\left. \frac{3}{2}x^2 - \frac{1}{3}x^3 \right|_0^3 = \left[\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3 \right] - \left[\frac{3}{2}(0)^2 - \frac{1}{3}(0)^3 \right] = \frac{27}{2} - 9 = \frac{9}{2} = 4.5$$



2. If $MR(q) = -q^2 + 10$, $MC(q) = q + 4$ and $FC = 0.5$ (assume q is in thousands, $TR/TC/FC$ in thousands and MR/MC are in dollars per item)

- (a) Find the profit if $q = 1$ thousand items are sold.

Answer:

$$P(1) = \int_0^1 MR(q) - MC(q) dq - FC = \int_0^1 (-q^2 + 10) - (q + 4) dq - 0.5$$

Simplifying and evaluating gives

$$\begin{aligned} P(1) &= \int_0^1 -q^2 - q + 6 dq - 0.5 = \left. -\frac{1}{3}q^3 - \frac{1}{2}q^2 + 6q \right|_0^1 - 0.5 \\ &= \left[-\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 + 6(1) \right] - \left[-\frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 + 6(0) \right] - 0.5 = -\frac{1}{3} - \frac{1}{2} + 6 - 0.5 = \frac{15}{3} \end{aligned}$$

Thus, $P(1) = 4.667$ thousand dollars.

- (b) Find the maximum profit.

Answer:

The maximum profit occurs when $MR(q) = MC(q)$. Solving gives $-q^2 + 10 = q + 4$, so $0 = q^2 + q - 6 = (q + 3)(q - 2)$. Thus, the solutions are $q = 2$ and $q = -3$. So max profit must occur at $q = 2$.

Now we compute the maximum profit, $P(2)$.

$$P(2) = \int_0^2 MR(q) - MC(q) dq - FC = \int_0^2 (-q^2 + 10) - (q + 4) dq - 0.5$$

Simplifying and evaluating gives

$$\begin{aligned} P(2) &= \int_0^2 -q^2 - q + 6 dq - 0.5 = \left. -\frac{1}{3}q^3 - \frac{1}{2}q^2 + 6q \right|_0^2 - 0.5 \\ &= \left[-\frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 + 6(2) \right] - \left[-\frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 + 6(0) \right] - 0.5 = -\frac{8}{3} - 2 + 12 - 0.5 = \frac{41}{6} \end{aligned}$$

Thus, $P(2) = 6.833$ thousand dollars.