12.4 Overview

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections.

12.4: Indefinite Integral Applications

We already know the connections between antiderivatives and their derivatives. The new fact is that we can actually find the formula for the original function if we start with the derivative (i.e. rate) information. So first, let me summary (for at least the third time in my review sheets) the connections between functions and their derivatives:

ORIGINAL ($f(x)$)	DERIVATIVE ($f'(x)$)	SECOND DERIVATIVE ($f''(x)$)
$f(x)$ = height of original at x	$f'(x)$ = slope of original at x	
increasing (uphill left-to-right)	positive (above x-axis)	
decreasing (downhill left-to-right)	negative (below x-axis)	
horizontal tangent	zero (crosses x-axis)	
concave up		positive
concave down		negative
possible inflection point		zero

If I haven't said it enough, here it is again: These are the fundamental facts that we use over and over again in every application problem. Okay, now for the new stuff.

How to find the '$+C$':

If you start with a derivative and you want to find the antiderivative (original) function, then you know from the previous section that it will have a '$+C$' at the end because without any more information you don't know how high or low to draw the graph (shifting the graph up or down does not change the slope). So in order to find the original function and determine the constant of integration C, you need to be given some information about the original function. This is called the *initial condition.* Let's see some examples:

- If $f'(x) = 12x^2 - 2$ and $f(1) = 5$, then find the function $f(x)$. (This question is asking you to find the general antiderivative of $12x^2 - 2$, then use the initial condition that $f(1) = 5$ to find C.).

  1. Finding the general antiderivative:
  $$\int 12x^2 - 2\,dx = 12\frac{1}{3}x^3 - 2x + C = 4x^3 - 2x + C = f(x).$$
  2. Use the initial condition to find C: Since $f(1) = 5$, when $x = 1$ the output is 5, so we get
  $$4(1)^3 - 2(1) + C = 5 \quad \text{which gives} \quad C = 3.$$
  And the final answer is
  $$f(x) = 4x^3 - 2x + 3.$$
  3. Check your work:
     - Derivative Check: $f'(x) = 12x^2 - 2$ (that matches!).
     - Initial Condition Check: $f(1) = 4(1)^3 - 2(1) + 3 = 5$ (matches, we know we are right!).

- If $g'(x) = 10$ and $g(4) = 7$, then find the function $g(x)$.
  1. Finding the general antiderivative:
  $$\int 10\,dx = 10x + C = g(x).$$
  2. Use the initial condition to find C: Since $g(4) = 7$, when $x = 4$ the output is 7, so we get
  $$10(4) + C = 7 \quad \text{which gives} \quad C = -33.$$
  And the final answer is
  $$g(x) = 10x - 33.$$
  3. Check your work:
     - Derivative Check: $g'(x) = 10$ (that matches!).
     - Initial Condition Check: $g(4) = 10(4) - 33 = 7$ (matches, we know we are right!).
- If $h'(x) = \sqrt{x} - \frac{1}{x}$ and $h(1) = 6$, then find the function $h(x)$.
  1. Finding the general antiderivative:
  $$\int x^{1/2} - \frac{1}{x}\,dx = \frac{1}{3/2}x^{3/2} - \ln(x) + C = \frac{2}{3}x^{3/2} - \ln(x) + C = h(x).$$
  2. Use the initial condition to find C: Since $h(1) = 6$, when $x = 1$ the output is 6, so we get
  $$\frac{2}{3}(1)^{3/2} - \ln(1) + C = 6 \quad \text{which gives} \quad C = 6 - \frac{2}{3} = \frac{16}{3}.$$
  And the final answer is
  $$h(x) = \frac{2}{3}x^{3/2} - \ln(x) + \frac{16}{3}.$$
  3. Check your work:
     - Derivative Check: $h'(x) = x^{1/2} - \frac{1}{x}$ (that matches!).
     - Initial Condition Check: $h(1) = \frac{2}{3}(1)^{3/2} - \ln(1) + \frac{16}{3} = \frac{18}{3} = 6$ (matches, we know we are right!).
- If $C'(x) = e^{-2x} + 4x + x^3$ and $C(0) = 10$, then find the function $C(x)$.
  1. Finding the general antiderivative (note that I used K to avoid confusion with the C from the function name):
  $$\int e^{-2x} + 4x + x^3\,dx = \frac{1}{-2}e^{-2x} + 4 \cdot \frac{1}{2}x^2 + \frac{1}{4}x^4 + K = -\frac{1}{2}e^{-2x} + 2x^2 + \frac{1}{4}x^4 + K = C(x).$$
  2. Use the initial condition to find K: Since $C(0) = 10$, when $x = 0$ the output is 10, so we get
  $$-\frac{1}{2}e^{-2(0)} + 2(0)^2 + \frac{1}{4}(0)^4 + K = 10 \quad \text{which gives} \quad K = 10 + \frac{1}{2} = \frac{21}{2}.$$
  And the final answer is
  $$C(x) = -\frac{1}{2}e^{-2x} + 2x^2 + \frac{1}{4}x^4 + \frac{21}{2}.$$
  3. Check your work:
     - Derivative Check: $C'(x) = e^{-2x} + 4x + x^3$ (that matches!).
     - Initial Condition Check: $C(0) = -\frac{1}{2}e^{-2(0)} + 2(0)^2 + \frac{1}{4}(0)^4 + \frac{21}{2} = -\frac{1}{2} + \frac{21}{2} = 10$ (matches, we know we are right!).

Some Application Reminders

We already know the initial conditions for most of the business functions. Remember that:

1. $TR(0) = 0$ is always assume (if you sell zero items, you get zero for total revenue).

2. $VC(0) = 0$ is always assume (if you produce zero items, then there is no variable cost).

3. $TC(0) = FC$ (you have to be given the fixed cost or some other initial information to find the formula for $TC(x)$).

4. $P(0) = -FC$ (you have to be given the fixed cost or some other initial information to find the formula for profit $P(x)$.)

5. Remember that $TC(x) = VC(x) + FC$ and $P(x) = TR(x) - TC(x)$.

6. Also remember the general definitions:

Totals	Marginals (derivatives)	Averages
$TR(x) = \int MR(x)\,dx$	$MR(x) = TR'(x)$	$AR(x) = \frac{TR(x)}{x} = p(x) =$ price
$TC(x) = \int MC(x)\,dx$	$MC(x) = TC'(x)$	$AC(x) = \frac{TC(x)}{x}$, so $TC(x) = xAC(x)$
$VC(x) = \int MC(x)\,dx$	$MC(x) = VC'(x)$	$AVC(x) = \frac{VC(x)}{x}$, so $VC(x) = xAVC(x)$
$P(x) = \int MP(x)\,dx$	$MP(x) = P'(x)$	

Given an application problem. Here are some steps to complete the task:

1. *Read the question*: Identify and label all functions and values involved. Write down the connections you know (from above).

2. *What are you given and what do you want*: Write down on the first line what you are given. Are you studying the height? Are you studying MC? Are you studying the speed? Whatever you are studying put that on your first line that is your 'original' function for this problem). And on the last line write what you want and what your final answer will look like (Are you looking for height or a time or an interval of time or a formula?, You should immediately know what form your answer will take and you should make a note to yourself so that you don't forget to put your answer in this form.).

3. *Fill in details*: Find the other functions and values involved in the problem. If you need to do a derivative or antiderivative, then do it right away. Then remember there are essentially 5 types of questions we ask over and over and over again (go see my 10.3 review sheet for those 5 questions and how to answer them). If you are stuck, just draw the 1st and 2nd derivative number lines for whatever function you are studying. Once you have these number lines you can answer any question!

4. *Solve, finish, and present your answer*: Solve for anything you can solve for, write down words to help explain what you are doing and present your answer in a way that other people can understand and read it.

On the next page are several examples from old exams:

(*Old Exam Question*): Consider $f(t) = 0.02t^3 - 0.39t^2 + 1.32t + 8$, and $g'(t) = 0.096t - 0.48$.

1. Question: *Find all values of t at which the tangent line to $g(t)$ has slope 3.*

   Answer:
   What do you want: Want to solve for a time t when the slope of $g(t)$ is equal to 3.
   What are you given: You are given for the formula for $g'(t)$ which is the slope of $g(t)$.
   Solve: $0.096t - 0.48 = 3$ gives $t = 36.25$.

2. Question: *Find all values of t at which the graph of $f(t)$ has a horizontal tangent line.*

   Answer:
   What do you want: Want to find t when $f(t)$ has a horizontal tangent, which is the same at the times t when $f'(t) = 0$.
   What are you given: You are given for the formula for $f(t)$, so you need to take the derivative to find $f'(t)$.
   Solve: $f'(t) = 0.06t^2 - 0.78t + 1.32 = 0$ and using the quadratic formula gives
   $t = \frac{0.78 \pm \sqrt{0.78^2 - (4)(0.06)(1.32)}}{(2)(0.06)}$ which gives $t = 2$ and $t = 11$.

3. Question: *Is $f(t)$ concave up or concave down at $t = 7$?*

   Answer:
   What do you want: Want to know if $f(t)$ is concave up or down at $t = 7$ which is the same as asking if $f''(7)$ is positive or negative.
   What are you given: You are given for the formula for $f(t)$, so you need to find $f''(t)$ and plug in $t = 7$.
   Solve: $f''(t) = 0.12t - 0.78$, so $f''(7) = 0.12(7) - 0.78 = 0.06$ which is positive. So Concave Up!

4. Question: *Suppose we are told that $f(11) = g(11)$. Find the formula for $g(t)$.*

   Answer:
   What do you want: Want to find the formula for $g(t)$.
   What are you given: You are given the formula for $g'(t)$, so you will need to find the antiderivative and use the given initial condition to solve for C.
   Solve: $g(t) = \int 0.096t - 0.48dt = 0.048t^2 - 0.48t + C$. Since $f(11) = g(11)$, we can plug into f to find that $f(11) = 0.02(11)^3 - 0.39(11)^2 + 1.32(11) + 8 = 1.95$. So we know that $g(11) = f(11) = 1.95$. That is, when $x = 11$, the output from g is supposed to be 1.95. Now we plug that in and solve for C to get $0.048(11)^2 - 0.48(11) + C = 1.95$ and $C = 1.422$. For a final answer of $g(t) = 0.048t^2 - 0.48t + 1.422$.

   And we check our work: $g'(t) = 0.096t - 0.48$ and $g(11) = 0.048(11)^2 - 0.48(11) + 1.422 = 1.95$ (so we know we are right!).

5. Question: *Which is smaller: A. The lowest value of $g(t)$ on the interval $t = 0$ to $t = 12$ OR B. The lowest value of $f(t)$ on the interval $t = 0$ to $t = 12$.*

   Answer:
   What do you want: Want to find which function $f(t)$ or $g(t)$ has the lower global minimum on the given interval.
   What are you given: You are given $f(t)$ and $g'(t)$, and you need to start by solving when $f'(t) = 0$ and when $g'(t) = 0$. Then find their global minimum (plugging in critical points and endpoints to the original functions).
   (Continues on next page)

Global Min for $f(t)$: Solving $f'(t) = 0$ gives $t = 2$ and $t = 11$ from an earlier part. So now we plug in the critical numbers and endpoints.
$f(0) = 8$, $f(2) = 0.02(2)^3 - 0.39(2)^2 + 1.32(2) + 8 = 9.24$, $f(11) = 0.02(11)^3 - 0.39(11)^2 + 1.32(11) + 8 = 1.95$, $f(12) = 0.02(12)^3 - 0.39(12)^2 + 1.32(12) + 8 = 2.24$.
Thus, the global minimum of $f(t)$ is 2.24 on this interval.
Global Min for $g(t)$: Solving $g'(t) = 0$ gives $0.096t - 0.48 = 0$, so $t = 5$. So now we plug in the critical numbers and endpoints.
$g(0) = 1.422$, $g(5) = 0.048(5)^2 - 0.48(5) + 1.422 = 0.222$, $g(12) = 0.048(12)^2 - 0.48(12) + 1.422 = 2.574$,
Thus, the global minimum of $g(t)$ is 0.222 on this interval.
Therefore g has a lower global minimum on this interval!

(Old Exam Question): *You sell Items. Your marginal revenue and marginal cost (both in dollars per Item) are given by the formulas* $MR(q) = 200 - 5.46q$ *and* $MC(q) = 3q^2 - 48q + 197$, *where* q *is in thousands of Items.* Note about units: This means that $q = 3.2$ is the same 3200 items. And the output from TR, TC, and P will be in thousands of dollars.

1. Question: *Find the quantity at which* TR *changes from increasing to decreasing.*

   Answer: *What do you want*: Want to know the quantity, q, where $TR(q)$ changes from increasing to decreasing, which is the same as asking when $MR(q)$ changes from positive to negative (meaning you need to know when $MR(q) = 0$!)
   What are you given: You are given for the formula for $MR(q)$, so you just need to set it equal to zero, solve and interpret.
   Solve: $MR(q) = 200 - 5.46q = 0$ gives $q = 36.630036$ which rounds to 36.630 thousand items. Note that $MR(q)$ changes from positive to negative at this value of q.

2. Question: *Find the quantity that maximizes profit.*

   Answer: *What do you want*: Want to know the quantity, q, where profit is maximized. So you need to know when we switch from $MR > MC$ to $MR < MC$ (i.e. you need to solve when $MR = MC$).
   What are you given: You are given for the formula for $MR(q)$ and $MC(q)$, so you just need to set them equal, solve and interpret.
   Solve: $200 - 5.46q = 3q^2 - 48q + 197$ gives $3q^2 - 42.54q - 3 = 0$ and using the quadratic formula we get
   $q = \frac{42.54 \pm \sqrt{42.54^2 - (4)(3)(-3)}}{2(3)}$ which gives $q = -0.070$ and $q = 14.250$ thousand items. Note that if you sketch a graph of MR and MC that at $q = 14.250$ we change from $MR > MC$ to $MR < MC$, so 14.250 is the desired quantity.

3. Question: *Give the formulas for total revenue and variable cost.*

   Answer: *What do you want*: Want TR and VC functions.
   What are you given: You are given for the formulas for $MR(q)$ and $MC(q)$ so you need to find the general antiderivatives and use the fact that $TR(0) = 0$ and $VC(0) = 0$ to find the constants for integration.
   Solve: $TR(q) = \int 200 - 5.46q dq = 200q - 2.73q^2 + C$, and since $TR(0) = 0$, we have $C = 0$, thus $TR(q) = 200q - 2.73q^2$.
   $VC(q) = \int 3q^2 - 48q + 197 dq = q^3 - 24q^2 + 197q + C$, and since $VC(0) = 0$, we have $C = 0$, thus $VC(q) = q^3 - 24q^2 + 197q$.