

Math 111  
University of Washington  
Graphical Approaches to Rates of Change

# 1 Introduction to Speed and Rates of Change

Speed is an example of a **rate of change**. You may be familiar with the formula

$$\text{speed} = \frac{\text{distance}}{\text{time}}.$$

Speed is the change in distance per unit time. That is, speed is the rate at which distance changes. We measure speed in units like miles per hour or feet per second or inches per year, and so forth.

There are two types of rate of change you will encounter in this course: overall and incremental. An **overall rate of change** is a rate of change measured from the starting time to a later time. An **incremental rate of change** is a rate of change measured between *any* two times.

In the context of speed, for example, if you drive 60 miles in the first 60 minutes of your trip, then your **average trip speed** at time  $t = 60$  minutes is 1 mile per minute. Since this speed is computed using the distance covered and time elapsed since the beginning of your trip, average trip speed is an example of an overall rate of change. On the other hand, if you time a mile in the middle of your trip and find that it takes 1.6 minutes, then your **average speed** over that period of time is

$$\frac{1 \text{ mile}}{1.6 \text{ minutes}} = \frac{1}{1.6} \frac{\text{mile}}{\text{minutes}} = 0.625 \text{ miles per minute.}$$

Since this average speed is not measured since the beginning of your trip, this is an incremental rate of change. (The **actual speed** of the car at any instant is an example of an **instantaneous rate of change**, which we will cover in Math 112.)

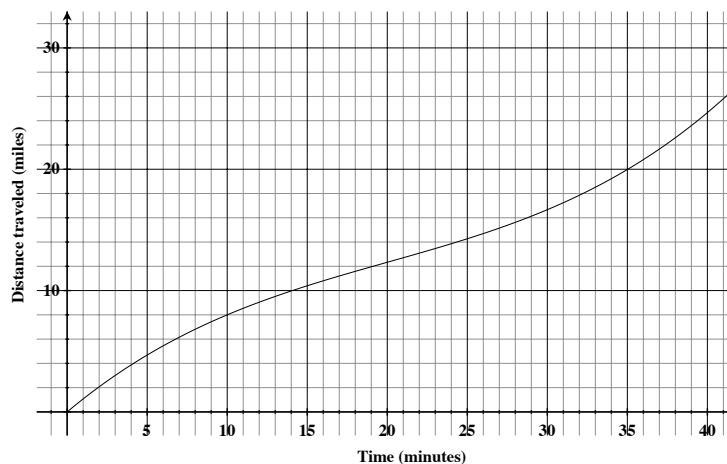
In general, by definition,

$$\text{average trip speed} = \frac{\text{total distance traveled so far}}{\text{total time elapsed so far}}$$

$$\text{average speed on an interval} = \frac{\text{distance traveled during that interval}}{\text{time elapsed during that interval}}.$$

## 2 The Graphical Interpretation of Average Trip Speed and Average Speed

The definitions of speed will get us only so far. There are instances in which it will be necessary to have a graphical representation of these quantities. At right is the graph of distance traveled vs. time for a moving car.

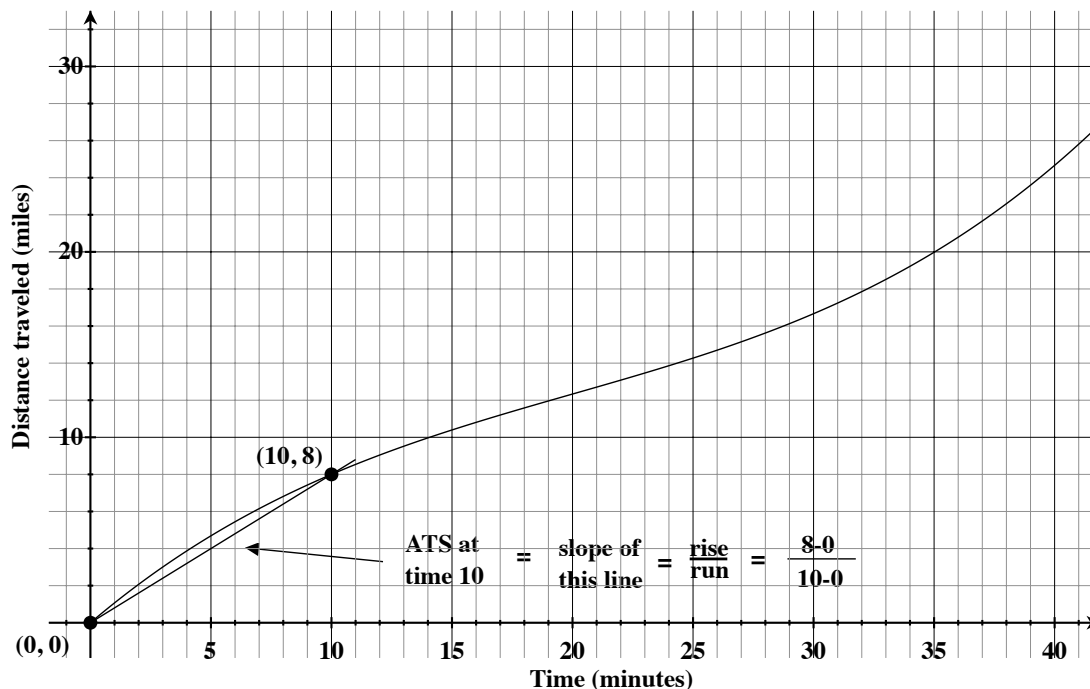


### 2.1 Average Trip Speed (ATS)

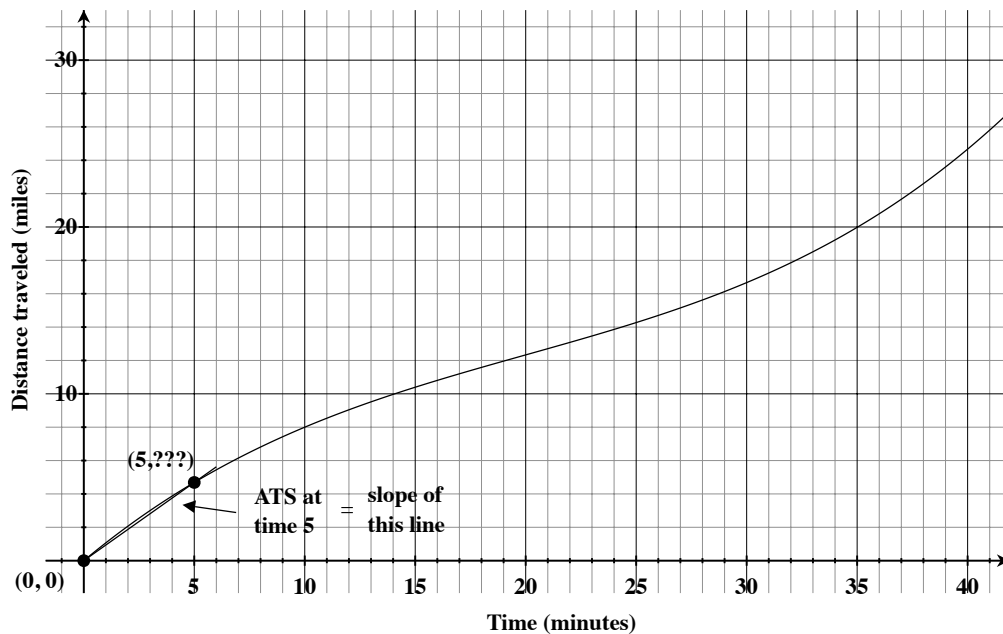
It's easy enough to use the definition of average trip speed to find ATS at  $t = 10$  minutes because it is clear from the graph that distance traveled at  $t = 10$  minutes is 8 miles. Then

$$\text{ATS at time } 10 = \frac{8 \text{ miles}}{10 \text{ minutes}} = 0.8 \text{ mpm.}$$

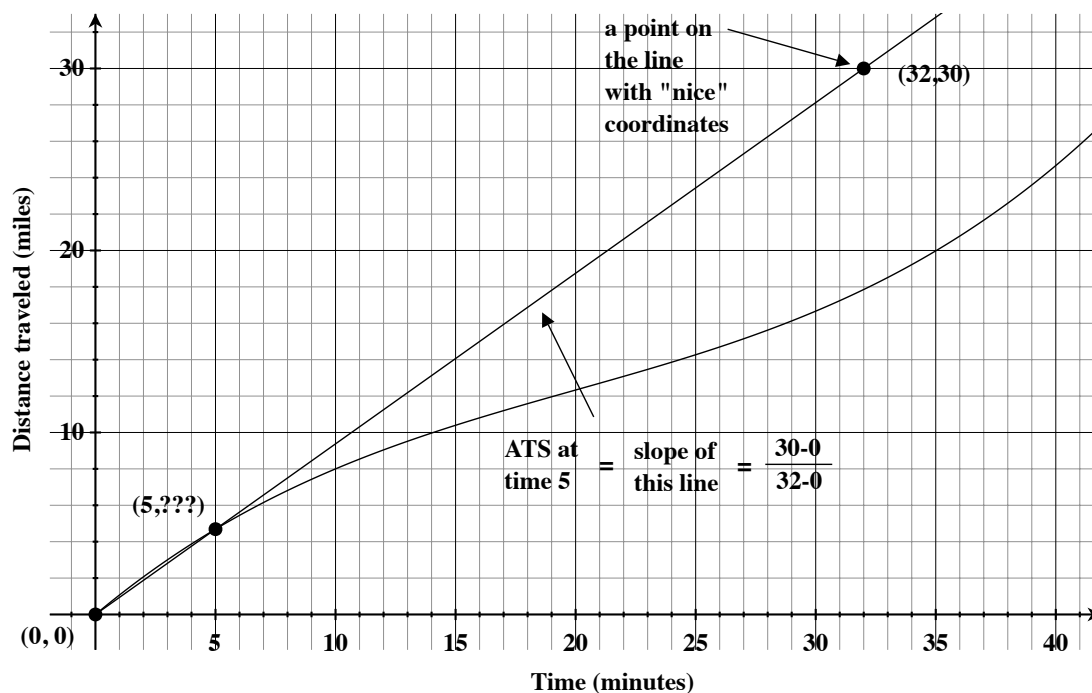
Notice that you can think of this speed as the slope of the line that goes through the origin  $(0,0)$  and the point  $(10,8)$ , the point through the distance graph at  $t = 10$ :



A line through the origin is called a **diagonal line**. So, we can think of ATS at  $t = 10$  as the slope of the diagonal line that intersects the distance graph at  $t = 10$ . Similarly, the ATS at  $t = 5$  is the slope of the diagonal line that intersects the distance graph at  $t = 5$ .



This time, however, the distance traveled at  $t = 5$  is not so obvious. In order to use the *definition* of average trip speed, we would have to approximate distance traveled at  $t = 5$ . We certainly could do this but we can minimize the error in our approximations by using the graphical interpretation of the ATS instead. If we extend the diagonal line as far as possible, we may be able to find a point on that line with “nice” coordinates. To compute the slope of the line, you may use *any* two points that sit on the line. That slope will be the ATS at  $t = 5$ :



Once we extend the line, we see that it includes the point (32, 30). Since it’s a diagonal line, it also contains the origin (0,0). We use the usual formula for the slope of a line between two points to find:

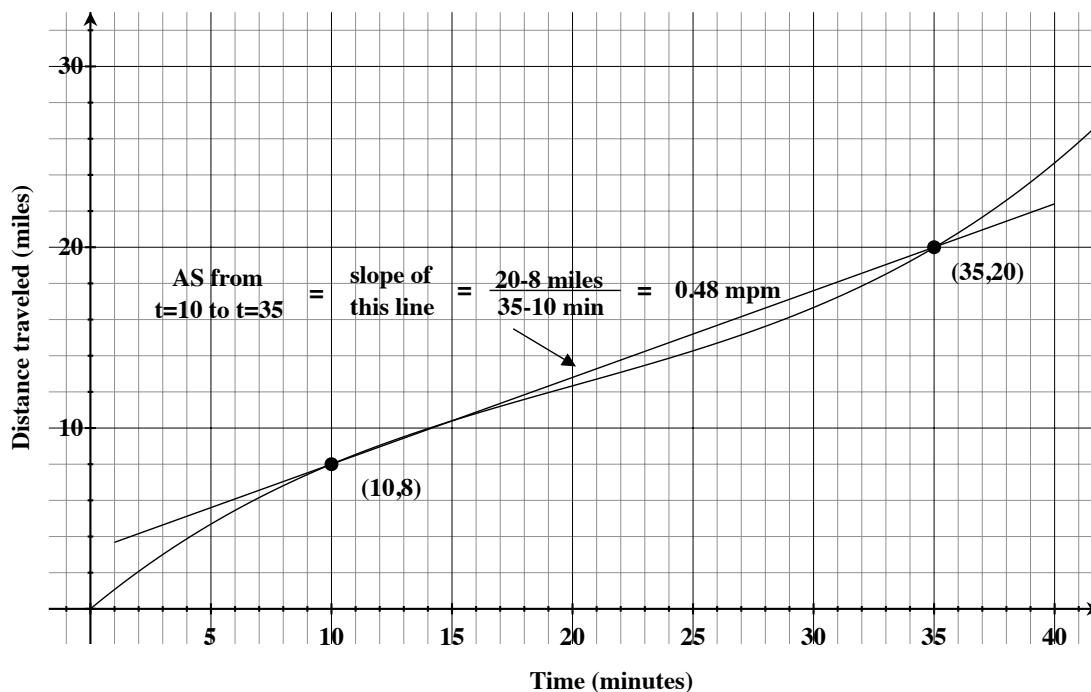
$$\text{ATS at time 5} = \frac{30 - 0}{32 - 0} = 0.9375 \text{ mpm.}$$

## 2.2 Average Speed (AS)

We can use the idea of a slope to develop a graphical interpretation of average speed on an interval. For example, we can use the graph and the definition of average speed to compute the average speed from  $t = 10$  to  $t = 35$ . The graph shows that the distance traveled at  $t = 10$  minutes is 8 miles and, at  $t = 35$  minutes, distance traveled is 20 miles. Between these two times, the car travels 12 miles and the time elapsed is 25 minutes. So,

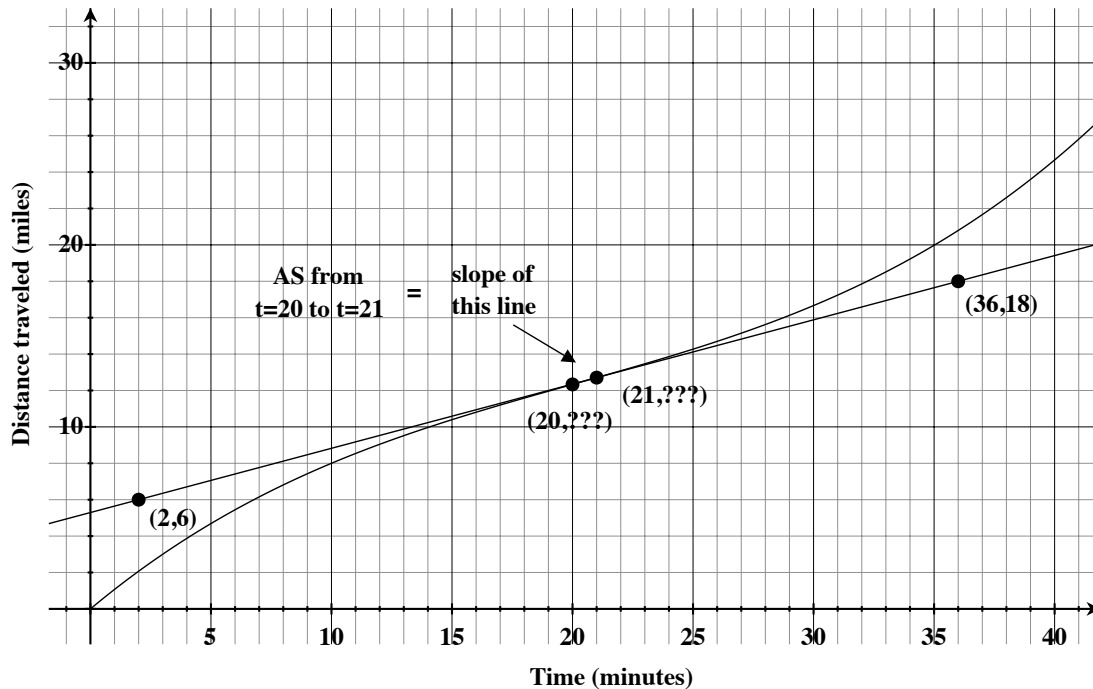
$$\text{AS from 10 to 35 min} = \frac{12 \text{ miles}}{25 \text{ minutes}} = 0.48 \text{ mpm.}$$

Notice that this is simply the slope of the line through the distance graph at  $t = 10$  and  $t = 35$ :



A line through two specific points on a graph is called a **secant line**. In general, the average speed from time  $a$  to time  $b$  is the slope of the secant line through the distance graph at  $t = a$  and  $t = b$ .

We'll use this idea to compute the average speed from  $t = 20$  to  $t = 21$ . The distance traveled between these two times is not obvious from the graph. So, we draw the secant line through the distance graph at  $t = 20$  and  $t = 21$ , extending it as far as possible in both directions, and look for two other points on this line with nice coordinates.



We find that  $(2, 6)$  and  $(36, 18)$  both sit on the line. So,

$$\text{AS from 20 to 21 min} = \frac{18 - 6}{36 - 2} = \frac{12}{34} \approx 0.3529 \text{ mpm.}$$

To summarize:

- To find average trip speed at a given time  $t$ , draw the *diagonal line* that intersects the distance graph at time  $t$ , extending it as far as possible. Look for two points with nice coordinates that sit on this line and use those coordinates to find the slope. (Note that, since this is a diagonal line, the origin  $(0, 0)$  should be one of the points you choose!)
- To find average speed from  $t = a$  to  $t = b$ , draw the *secant line* through the distance graph at  $t = a$  and  $t = b$ , extending it as far as possible in both directions. Look for two points with nice coordinates that sit on this line and use those coordinates to find the slope.

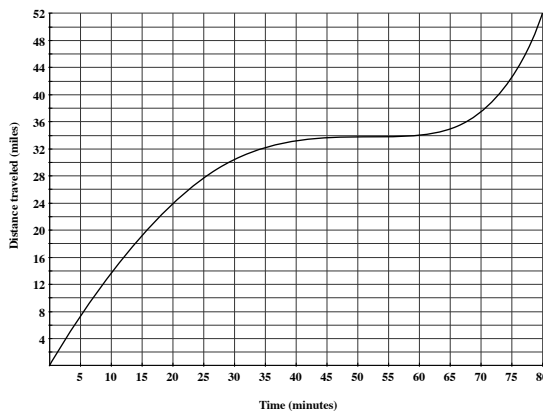
If you can't find two points with good coordinates, then you will be forced to do some approximating. If this is the case, choose two points on your line that are *far apart* and approximate them as carefully as you can. Choosing points that are far apart helps minimize the error in your approximations by giving a large denominator in your slope calculation.

### 3 Going Backwards

#### 3.1 Average Trip Speed

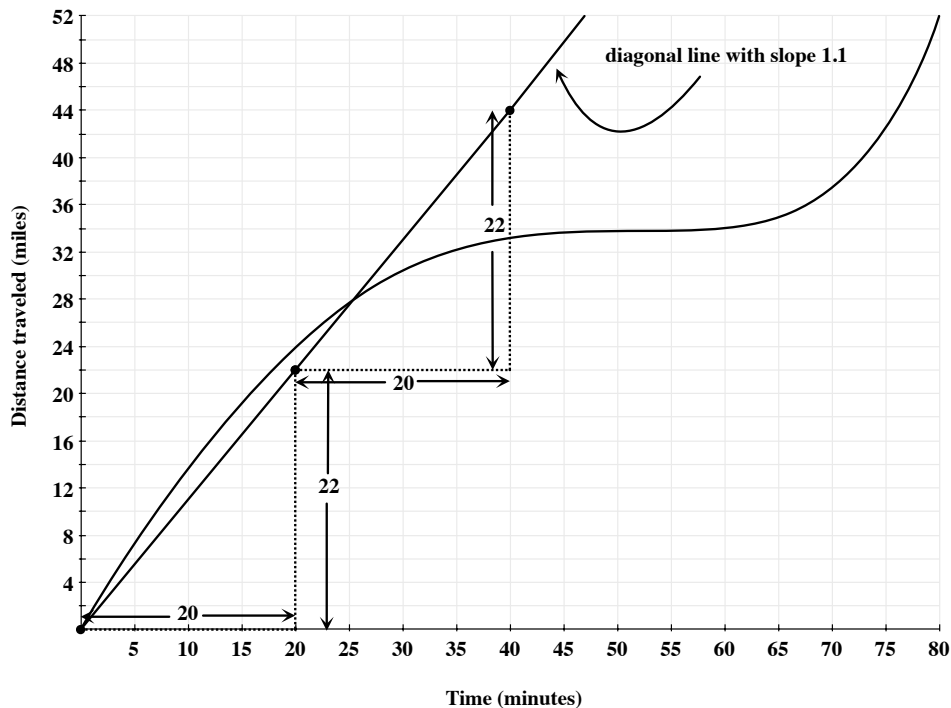
Recall the graphical interpretation of average trip speed:  $ATS$  at time  $t$  is the slope of the diagonal line through the distance graph at time  $t$ . If we are asked for the  $ATS$  at, for example,  $t = 10$ , we draw the diagonal line through the distance graph at  $t = 10$ , choose two convenient points on the line, and compute its slope. However, if we are *given* an average trip speed and asked when the car achieves that average trip speed, we must reverse the process: we draw a diagonal line that has the given slope and find the time at which it intersects the distance graph.

For example, suppose we are given the following distance graph and we want to find the time at which the average trip speed is 1.1 mpm. We need to draw a diagonal line with slope 1.1 and find the time at which this line intersects the distance graph. To do this, recall that slope is rise/run. A slope of 1.1 means a vertical change of 1.1 units for every 1-unit horizontal change. Since we cannot accurately mark a vertical change of 1.1 units on this graph, we do some numerical manipulation to obtain a more user-friendly slope:



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{1.1}{1} \times \frac{10}{10} = \frac{11}{10} \times \frac{2}{2} = \frac{22}{20}.$$

To draw a diagonal line with slope 1.1: mark the origin (since this is a diagonal line, that's one point this line will go through); go over 20 and up 22, marking the point (20, 22); for more accuracy, you can go over 20 and up 22 again and mark the point (40, 44); and draw the line through these three points.



Finally, to find the time at which average trip speed is 1.1 mpm, simply note the time at which this line intersects the distance graph:  $t \approx 25$  minutes.

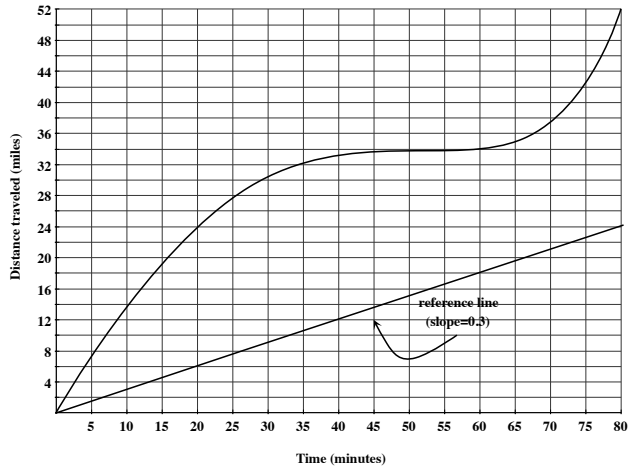
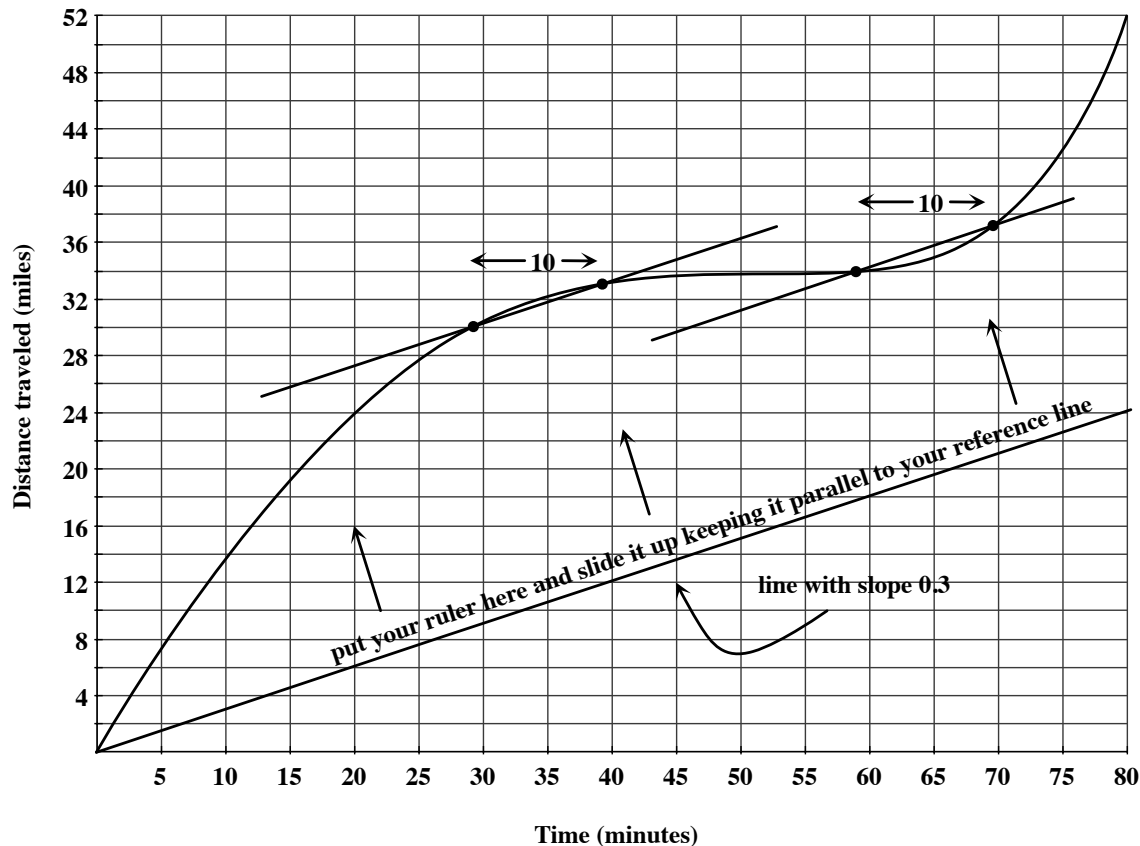
### 3.2 Average Speed

Backward questions involving average speed between two points is a bit more difficult. Recall the graphical interpretation of average speed from  $t = a$  to  $t = b$ : average speed from  $a$  to  $b$  is the slope of the secant line through the distance graph at  $t = a$  and  $t = b$ .

Using the same distance graph, we'll do an example in which we find a 10-minute interval during which the average speed is 0.3 mpm. We will need to find two times, ten minutes apart, such that the secant line through the distance graph at those two times has slope 0.3. It will help simply to see what *any line* with slope 0.3 looks like on this graph. So we draw a diagonal line with slope 0.3 using the same technique as above.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{0.3}{1} \times \frac{10}{10} = \frac{3}{10} \times \frac{2}{2} = \frac{6}{20}.$$

Start at the origin, go over 20 and up 6 a couple of times and draw a line with that slope. This will be our **reference line**. Place your ruler along this reference line and slide the ruler up, maintaining a slope of 0.3, until it intersects the distance graph at two points that are ten minutes apart. Note that there are two possible intervals to find:





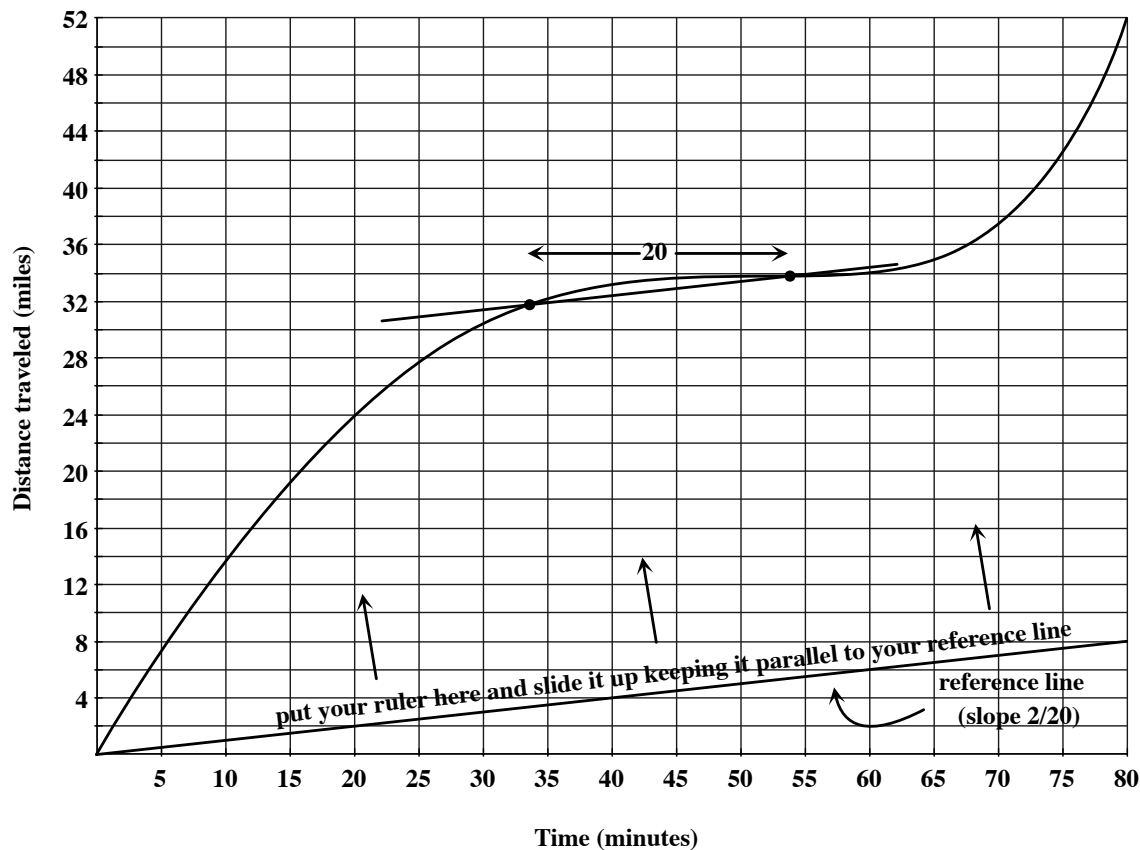
A ten-minute interval during which the average speed is 0.3 mpm is from  $t = 29$  to  $t = 39$  minutes. Another such interval is from  $t = 59$  to  $t = 69$  minutes.

### 3.3 Change in Distance

Using the same distance graph, we'll do an example in which we find a 20-minute interval during which the car travels 2 miles. Notice that, if the car travels 2 miles in 20 minutes, then the average speed of the car during that interval is given by

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{2 \text{ miles}}{20 \text{ minutes}}.$$

Recall that average speed over an interval is given by the slope of a secant line through the distance graph. In other words, we're looking for two times, twenty minutes apart, such that the secant line through the distance graph at those two times has slope  $\frac{2}{20}$ . We follow the procedure we just learned to find that the interval from  $t = 33$  to  $t = 53$  minutes, the car travels 2 miles. (Is there another such twenty-minute interval?)



## 4 Other Rates of Change

*Speed* is the rate of change of *distance*:

$$\text{speed} = \frac{\text{change in distance}}{\text{change in time}}.$$

In general, the rate of change of Something is:

$$\frac{\text{change in Something}}{\text{change in time}}.$$

### 4.1 Overall Rates of Change

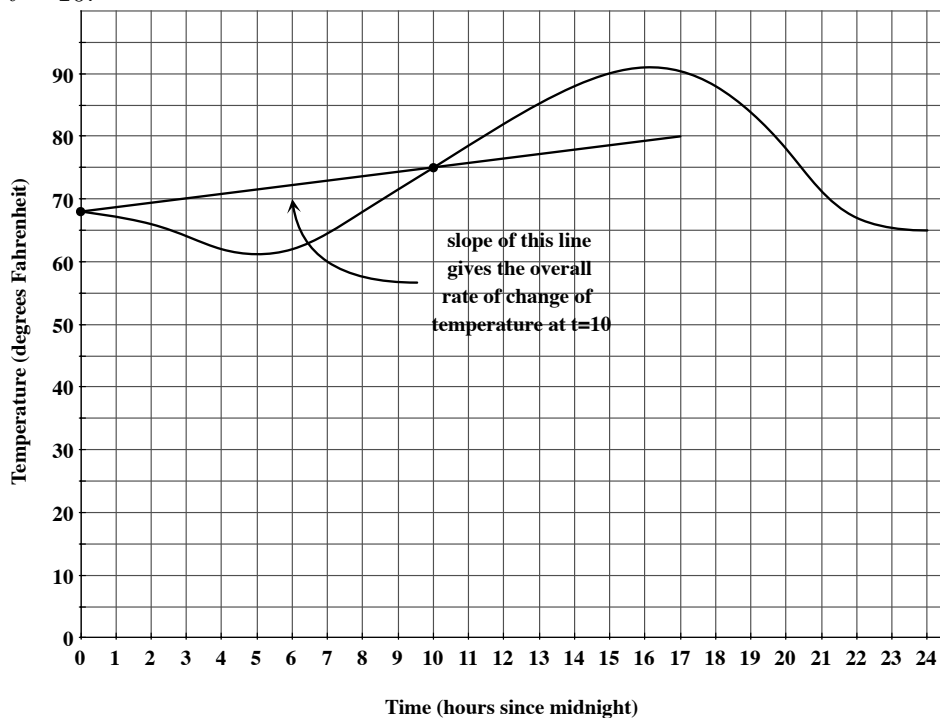
Average trip speed is an example of an **overall rate of change**: the rate at which distance has changed since  $t = 0$ . In general, the overall rate of change of Something is:

$$\frac{\text{change in Something since } t = 0}{\text{change in time since } t = 0}.$$

For example, the following is the graph of the temperature in Seattle on a recent (unusually hot) summer day. The overall rate of change of temperature at  $t = 10$  is

$$\frac{\text{change in temp from } t = 0 \text{ to } t = 10}{\text{change in time from } t = 0 \text{ to } t = 10} = \frac{75^\circ - 68^\circ}{10 \text{ hours}} = 0.7 \text{ degrees per hour.}$$

This is given by the slope of the secant line through the temperature graph at  $t = 0$  and  $t = 10$ .

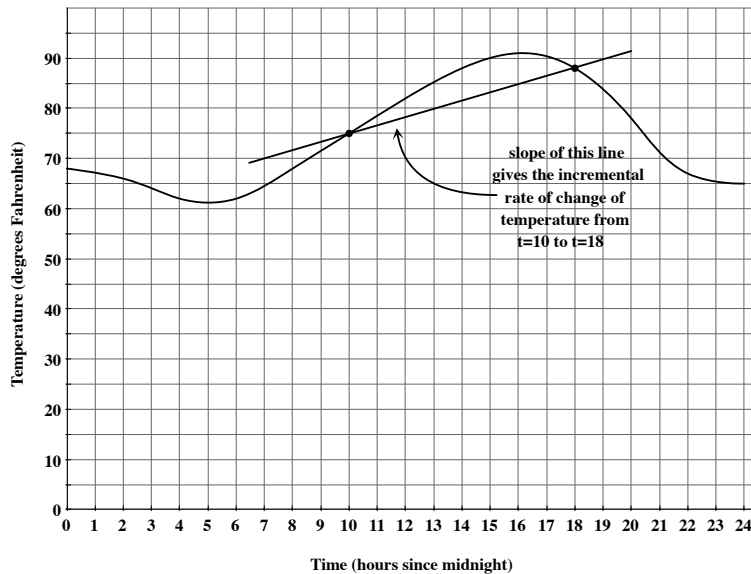


NOTE: The overall rate of change is given by the slope of a *secant line* through the graph at  $t = 0$  and some other time. If your graph goes through the origin (like the graph of distance traveled), then this secant line will go through the origin and will therefore also be a *diagonal line*.

## 4.2 Incremental Rates of Change

Average speed on an interval is an example of an **incremental rate of change**. Unlike an overall rate, which must begin at  $t = 0$ , an incremental rate may be computed between *any two times*. Just like average speed, an incremental rate of change is given by the slope of a secant line through a graph. In the temperature example, the incremental rate of change of temperature (or the **average change in temperature**) from  $t = 10$  to  $t = 18$  hours is

$$\frac{\text{change in temp from } t = 10 \text{ to } t = 18}{\text{change in time from } t = 10 \text{ to } t = 18} = \frac{88^\circ - 75^\circ}{8 \text{ hours}} = 1.625 \text{ degrees per hour.}$$



To summarize, given the graph of Something vs. time:

- The overall rate of change of Something at time  $t$  is

$$\frac{\text{change in Something from time 0 to time } t}{\text{change in time from time 0 to time } t}.$$

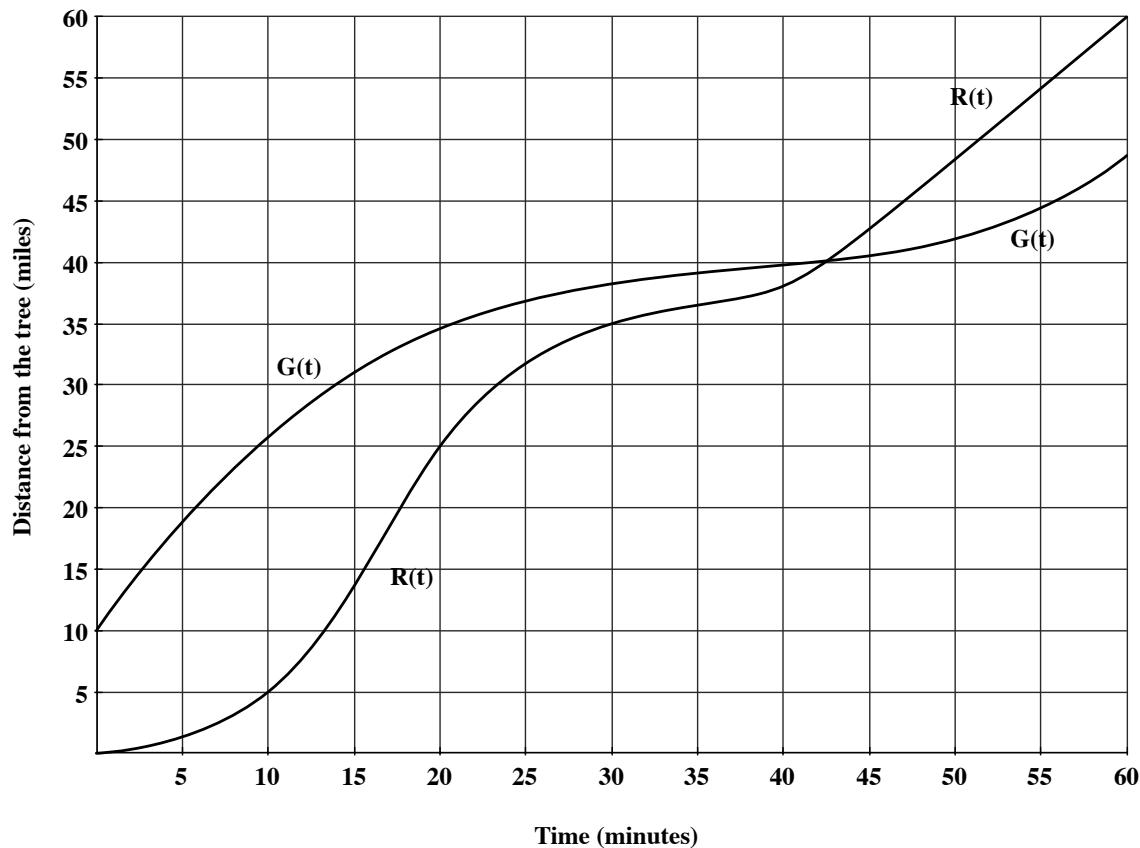
- The graphical interpretation of the overall rate of change at time  $t$  is the slope of the secant line through the Something graph at time 0 and time  $t$ . If the Something graph goes through the origin, then this will be the slope of the diagonal line through Something at time  $t$ .
- Average trip speed is an example of an overall rate of change. Average trip speed is the overall rate of change of distance traveled.
- The incremental rate of change of Something from  $t = a$  to  $t = b$  is

$$\frac{\text{change in Something from time } a \text{ to time } b}{\text{change in time from time } a \text{ to time } b}.$$

- The graphical interpretation of the average rate of change from  $t = a$  to  $t = b$  is the slope of the secant line through the Something graph at  $t = a$  and  $t = b$ .
- Average speed is an example of an incremental rate of change. Average speed is the incremental rate of change of distance traveled.

## 5 Three Languages

A red car and a green car travel down a long straight road. At  $t = 0$ , the red car passes a tree and the green car is 10 miles ahead of the red car. The following graph shows the distance from each car to the tree at various times during a 60 minute period.



Let  $R(t)$  denote the distance from the red car to the tree at time  $t$ . Then, for example,

$$R(10) = 5.$$

This is a statement in *functional notation*. It is a mathematical statement, read “ $R$  of 10 equals 5,” that has the following English translation:

The red car is 5 miles from the tree after 10 minutes.

We’ve labeled the red car’s distance graph  $R(t)$ . The graphical interpretation of the statement  $R(10) = 5$  is:

The height of the red car’s distance graph at  $t = 10$  is 5.

OR

The graph of  $R(t)$  goes through the point  $(10, 5)$ .

Similarly, we let  $G(t)$  represent the distance from the green car to the tree after  $t$  minutes. Then, for example,  $G(0) = 10$ . In English, this translates as:

The green car is 10 miles from the tree at  $t = 0$ .

Graphically, one interpretation would be:

The “ $y$ ”-intercept of the graph of  $G(t)$  is 10.

In this course, you're expected to read a question written in one of three languages: English, graphical interpretations, or functional notation. Often, you'll need to translate into one of the other languages in order to answer the question and then translate your answer back into the original language. It's important that you practice the skill of translating between these languages.

Here is a table of some common phrases in each language.

English	Graphical Interpretation	Functional Notation	
distance traveled by the red car from $t = 5$ to $t = 10$	change in height of the $R(t)$ graph from $t = 5$ to $t = 10$	$R(10) - R(5)$	
distance traveled by the red car in the first 20 minutes	the height of the $R(t)$ graph at $t = 20$	$R(20)$	*
distance traveled by the green car in the first 20 minutes	change in the height of the $G(t)$ graph from $t = 0$ to $t = 20$	$G(20) - G(0)$	*
average speed of the red car from 10 to 30 minutes	slope of the secant line through $R(t)$ at $t = 10$ and $t = 30$	$\frac{R(30) - R(10)}{30 - 10}$	
average trip speed of the red car after 40 minutes	slope of the <b>diagonal</b> line through $R(t)$ graph at $t = 40$	$\frac{R(40)}{40}$	**
average trip speed of the green car after 40 minutes	slope of the <b>secant</b> line through $G(t)$ graph at $t = 0$ and $t = 40$	$\frac{G(40) - G(0)}{40}$	**
_____	slope of the diagonal line through $G(t)$ graph at $t = 40$	$\frac{G(40)}{40}$	**
distance between the cars after 20 minutes	vertical distance between the distance graphs at $t = 20$	$G(20) - R(20)$	†
distance between the cars after 50 minutes	vertical distance between the distance graphs at $t = 50$	$R(50) - G(50)$	†
average speed of the red car during the $h$ -minute interval starting at $t = 5$	slope of the secant line through $R(t)$ graph at $t = 5$ and $t = 5 + h$	$\frac{R(5 + h) - R(5)}{h}$	‡
average speed of the red car during the 5-minute interval starting at time $t$	slope of the secant line through $R(t)$ graph at time $t$ and $t + 5$	$\frac{R(t + 5) - R(t)}{5}$	‡
the time at which both cars are the same distance from the tree	the time at which the distance graphs intersect	the time at which $R(t) = G(t)$	
The red car is 0 miles from the tree at $t = 0$ .	The graph of $R(t)$ goes through the origin.	$R(0)=0$	

You should get to the point where you can recognize the phrases in the above table in all three languages and translate easily between them in different contexts (not just distance and speed).

Here are some notes and questions for you to consider:

- The English translation should be about the physical story. Here, the physical story involves distance, time, and speed. In English, most numbers have units attached (miles, minutes, mpm, etc).
- The Graphical Interpretation should involve features of a graph: heights, intercepts, slopes, and vertical or horizontal changes, for example.
- Functional Notation may contain a few words of English. For example, the English question *when* may translate into Functional Notation as *for which value of t*:

**English:** When is the green car 5 miles ahead of the red car?

**Functional Notation:** For which value of  $t$  is  $G(t) = R(t) + 5$ ?

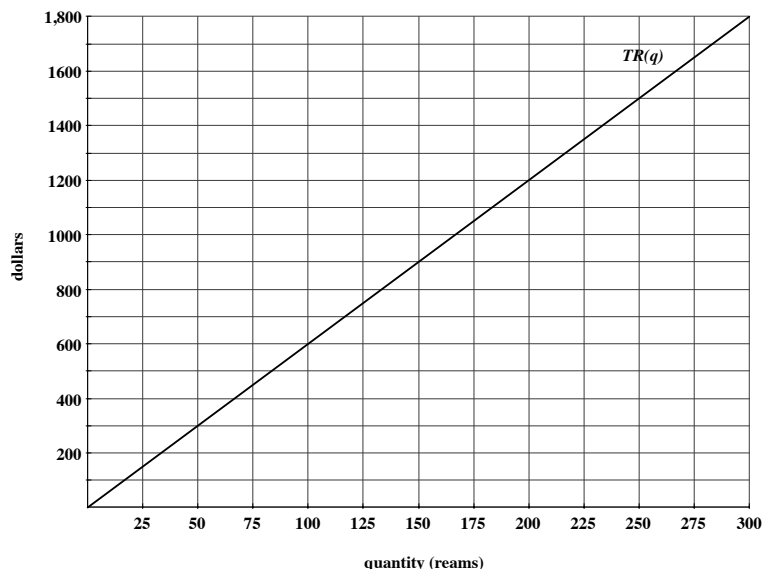
- There is one row in the table with no English translation—we can define all sorts of things mathematically and graphically that may have no practical meaning.
- The rows marked with a \* have markedly different graphical interpretations and functional notation translations even though the only difference in their English translations is the color of the car. What feature of the two distance graphs make this distinction necessary? This distinction will also explain the difference in the three rows marked with a \*\*.
- In the rows marked with a †, the order of the subtraction is important. Why do we change the order of the functions?
- Make sure you understand the distinction between the rows marked with a ‡. How else might these phrases be translated into English?

## 6 Introduction to Revenue, Cost, and Profit

Suppose we produce and sell goods. We'll use the word **cost** to refer to the money we *spend* during production; the word **revenue** will refer to the money we *receive* when we sell our products.

Suppose in particular, we own a print shop. We'll investigate our revenue and cost during one business day. Suppose we charge \$6 per ream for printing. If we sell 1 ream of printing, then we get \$6; if we sell 2 reams, we get \$12; etc. The amount of money that we receive for selling  $q$  reams is called our **total revenue**, which we denote  $TR(q)$  in functional notation. Note that, if we sell 0 reams, we receive 0 dollars. So,  $TR(0) = 0$ . This implies that the graph of  $TR$  goes through the origin. Note further that, every time we sell one more ream of printing, our  $TR$  goes up by the same amount: \$6. This means that the graph of  $TR$  will be a *line* with slope 6.

$q$ (reams)	$TR(q)$ (dollars)
0	0
1	6
2	12
3	18
50	300
100	600
150	900
200	1200
300	1800



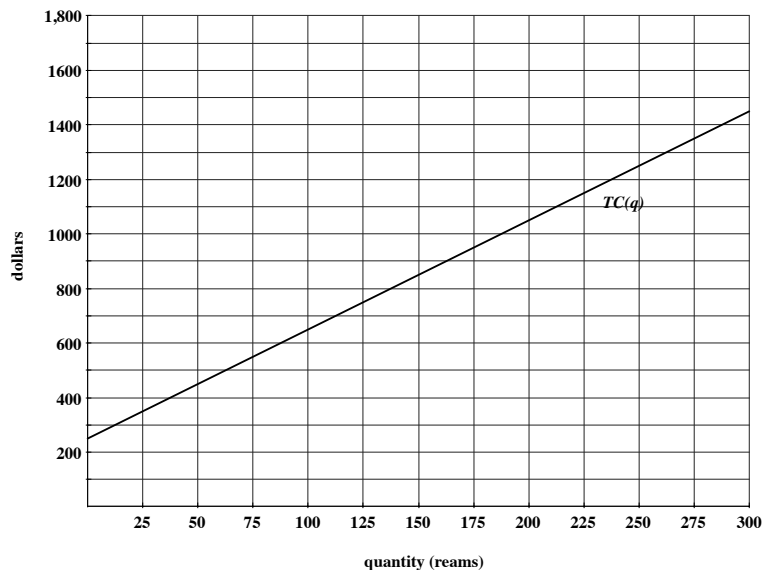
The *rate* at which total revenue changes (in dollars per ream) is called **marginal revenue**. In this example, total revenue always increases at the rate of \$6 per ream. So,  $MR = 6$ , the slope of the linear  $TR$  graph.

Costs are a little more complicated. There are costs associated with each ream of paper that we print: we pay for blank paper and ink, for example. These are called **variable costs**. They depend on the number of reams we actually produce. We use  $VC(q)$  to denote the variable cost to print  $q$  reams. Suppose it costs \$4 to print each ream. Then, for example,  $VC(0) = 0$ ,  $VC(100) = 400$ ,  $VC(200) = 800$ , etc. However, these are not the only costs we have. There are costs we must pay just for opening the shop for the day, whether or not we sell any printing. These are known as **fixed costs** (or *overhead*) and they include, for example, rent and some salaries and utilities. Fixed costs are denoted  $FC$ . Suppose we have \$250 in fixed costs during a single business day. Then, if we sell no printing, we still have \$250 in costs to pay. If we sell 100 reams of printing, we must pay \$250 in fixed costs plus the \$400 it costs to print 100 reams. The total amount of money we spend to print  $q$  reams of paper is our **total cost**, denoted  $TC(q)$ . Total cost is the sum of variable cost and fixed cost:

$$TC(q) = VC(q) + FC.$$

In this print shop, each ream costs \$4 to print and we have \$250 in fixed costs. The value of  $FC$  is the “ $y$ ”-intercept of the  $TC$  graph (since we still pay our fixed costs when we print 0 reams) and each time we print one more ream, total cost goes up by \$4. This means that the graph of  $TC(q)$  will be a line with “ $y$ ”-intercept 250 and slope 4.

$q$ (reams)	$VC(q)$ (dollars)	$TC(q)$ (dollars)
0	0	250
1	4	254
2	8	258
50	200	450
100	400	650
200	800	1050
300	1200	1450

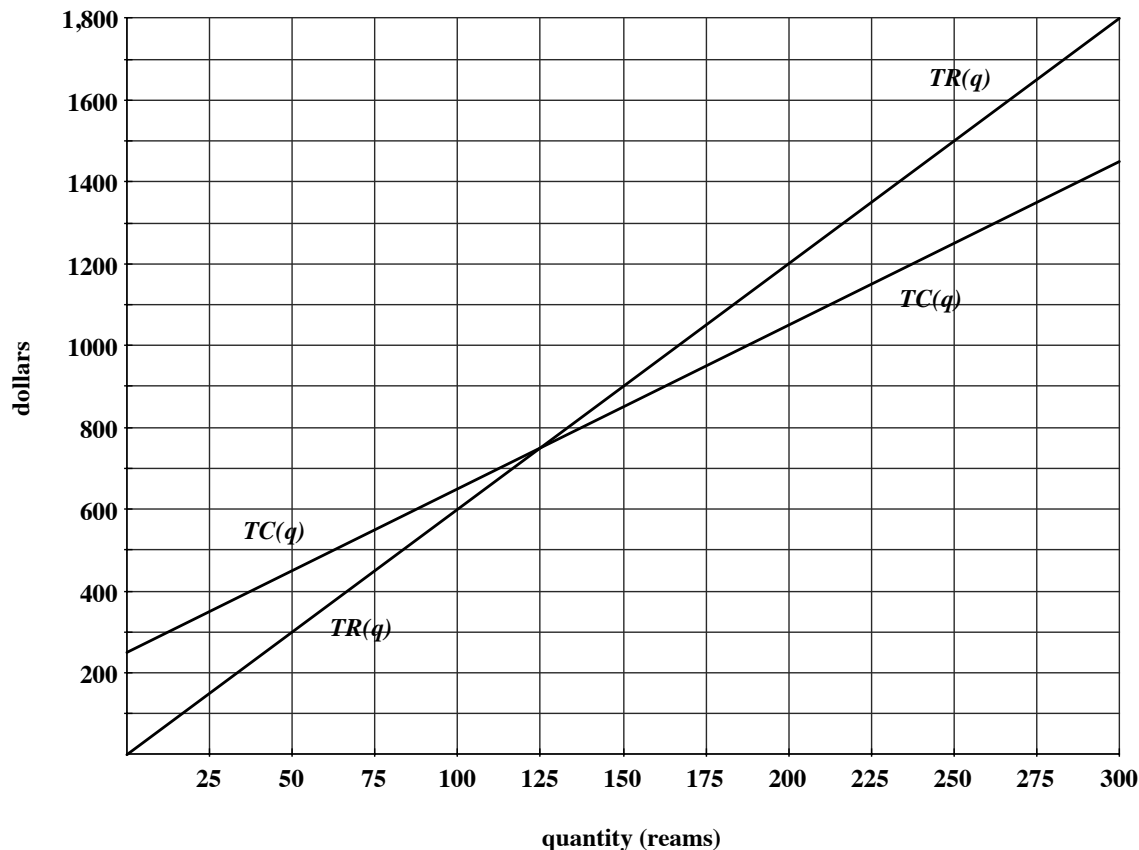


The *rate* at which total cost increases is called **marginal cost**, denoted  $MC$ . In this example, total cost increases at a rate of \$4 per ream. So,  $MC = 4$ , the slope of the linear  $TC$  graph.

If we subtract total cost from total revenue, we get profit, denoted  $P(q)$ . In functional notation,

$$P(q) = TR(q) - TC(q).$$

Profit may be positive, negative, or zero. To see how, look at the graphs of  $TR$  and  $TC$  on the same set of axes:



Profit is positive when total revenue exceeds total cost (i.e., when the graph of  $TR$  is above the graph of  $TC$ ). For example, if we print and sell 200 reams, total revenue is \$1200,



total cost is \$1050, and profit is \$150:

$$P(200) = TR(200) - TC(200) = \$1200 - \$1050 = \$150.$$

Profit is negative when total cost is larger than total revenue. For example,

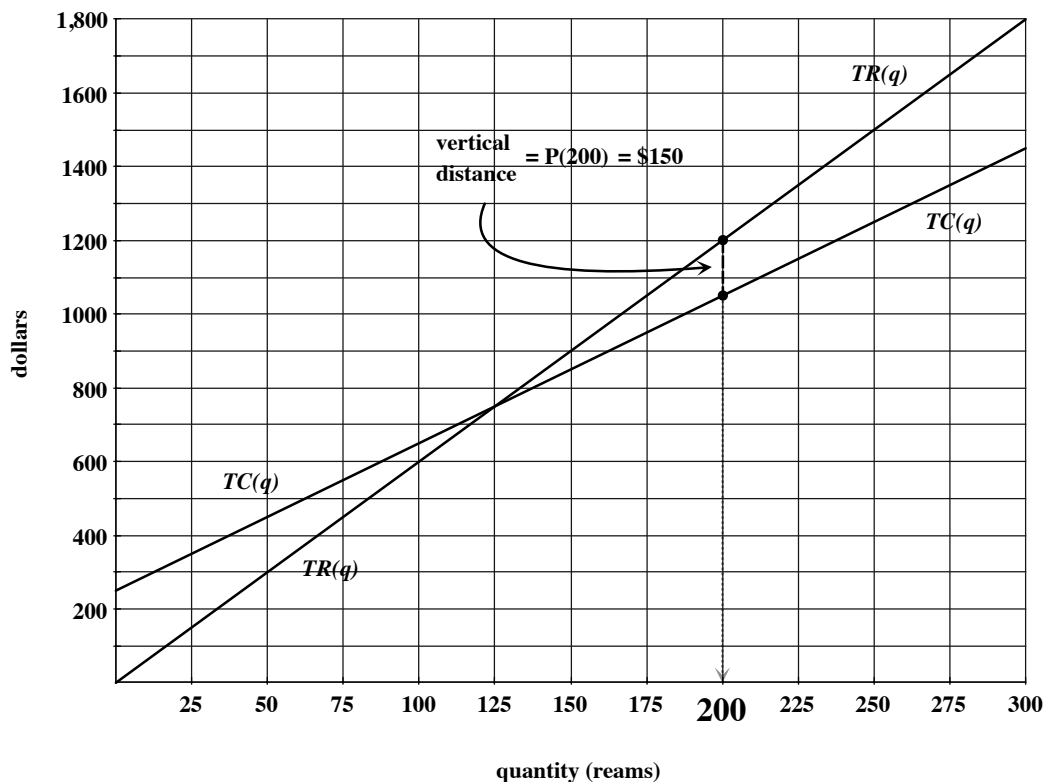
$$P(100) = TR(100) - TC(100) = \$600 - \$650 = -\$50.$$

Profit is zero when total revenue and total cost are equal. In our print shop, this occurs when we sell 125 reams:

$$P(125) = TR(125) - TC(125) = \$750 - \$750 = \$0.$$

(Colloquially, when profit is positive, people say they are *making a profit*; when profit is negative, they are *taking a loss*; and when profit is 0, they are *breaking even*.)

When profit is positive, the graphical interpretation of profit is the vertical gap between the  $TR$  and  $TC$  graphs.



Notice that, starting at  $q = 125$  reams, the graph of  $TR$  is above the graph of  $TC$  and the vertical distance between them gets larger as quantity increases. So, profit is increasing on the interval from  $q = 125$  to  $q = 300$  reams (and would continue to increase if we extended the horizontal axis even further).

To summarize:

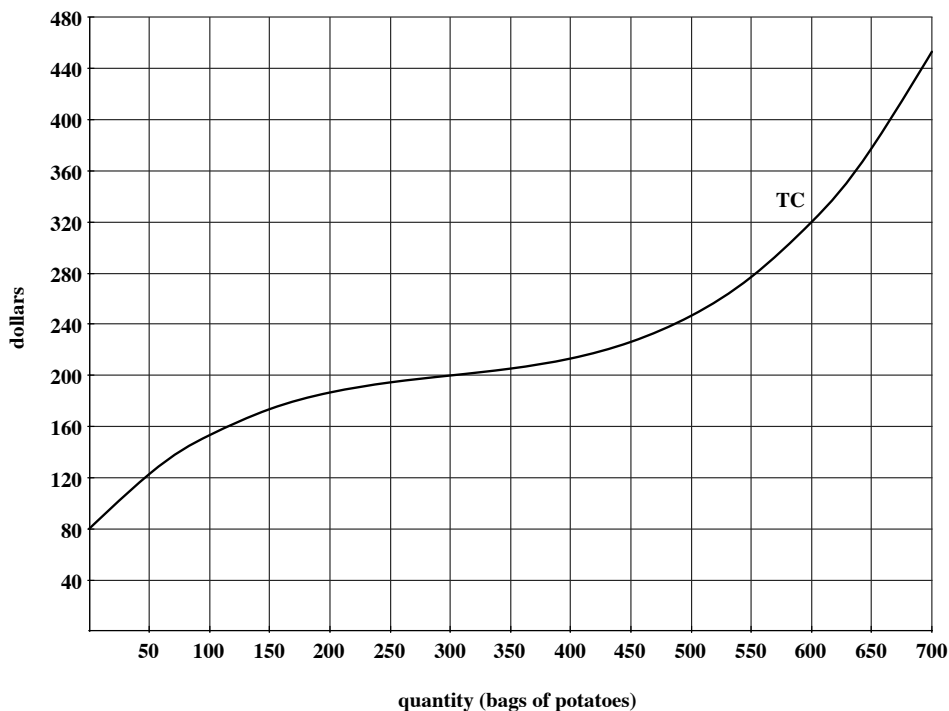
- Total revenue ( $TR$ ) is the amount of money you receive for selling a product.
- If you sell nothing, you receive no money. So,  $TR(0) = 0$  in every scenario. This means that the graph of  $TR$  will always go through the origin.
- Marginal revenue ( $MR$ ) is the rate at which  $TR$  changes. If  $TR$  is linear,  $MR$  is its slope.
- Total cost ( $TC$ ) is the amount of money you spend to produce your product.
- Fixed costs ( $FC$ ) are the costs you pay even if you don't produce any goods. In functional notation,  $FC = TC(0)$ . Graphically,  $FC$  is the "y"-intercept of the  $TC$  graph.
- Variable costs ( $VC$ ) are the costs that depend on the level of production.
- In functional notation,  $TC(q) = VC(q) + FC$ .
- Marginal cost ( $MC$ ) is the rate at which  $TC$  changes. If  $TC$  is linear, then  $MC$  is its slope.
- In functional notation, profit is given by  $P(q) = TR(q) - TC(q)$ .
- Graphically, at a quantity where the  $TR$  graph is above the  $TC$  graph, profit is given by the vertical distance between the  $TR$  and  $TC$  graphs.

## 7 Marginal Cost and Marginal Revenue

### 7.1 Marginal Cost as the Slope of a Secant Line

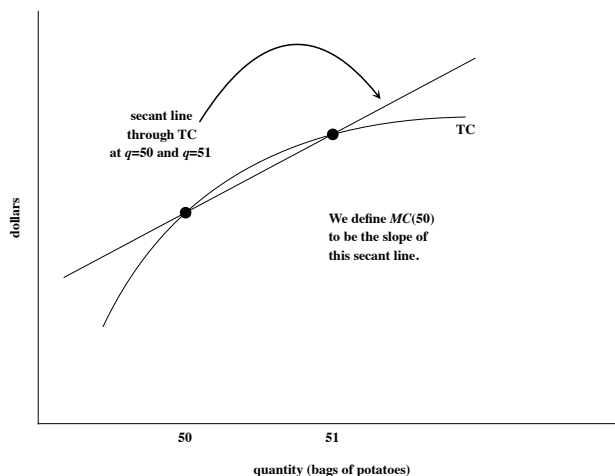
In the print shop from the previous example, both total revenue and total cost were linear. This need not be the case. For example, you may offer discounts for larger orders, meaning that the price per ream of printing would change depending on how many a customer purchases—this would cause total revenue to be non-linear. From the production side, things like overtime and machine efficiency may cause total cost to be non-linear. In this section, we'll investigate a non-linear total cost graph.

A farmer produces potatoes that will be shipped to a processor that makes french fries for fast food restaurants. The graph below shows the farmer's total cost at various production levels from 0 to 700 bags of potatoes.



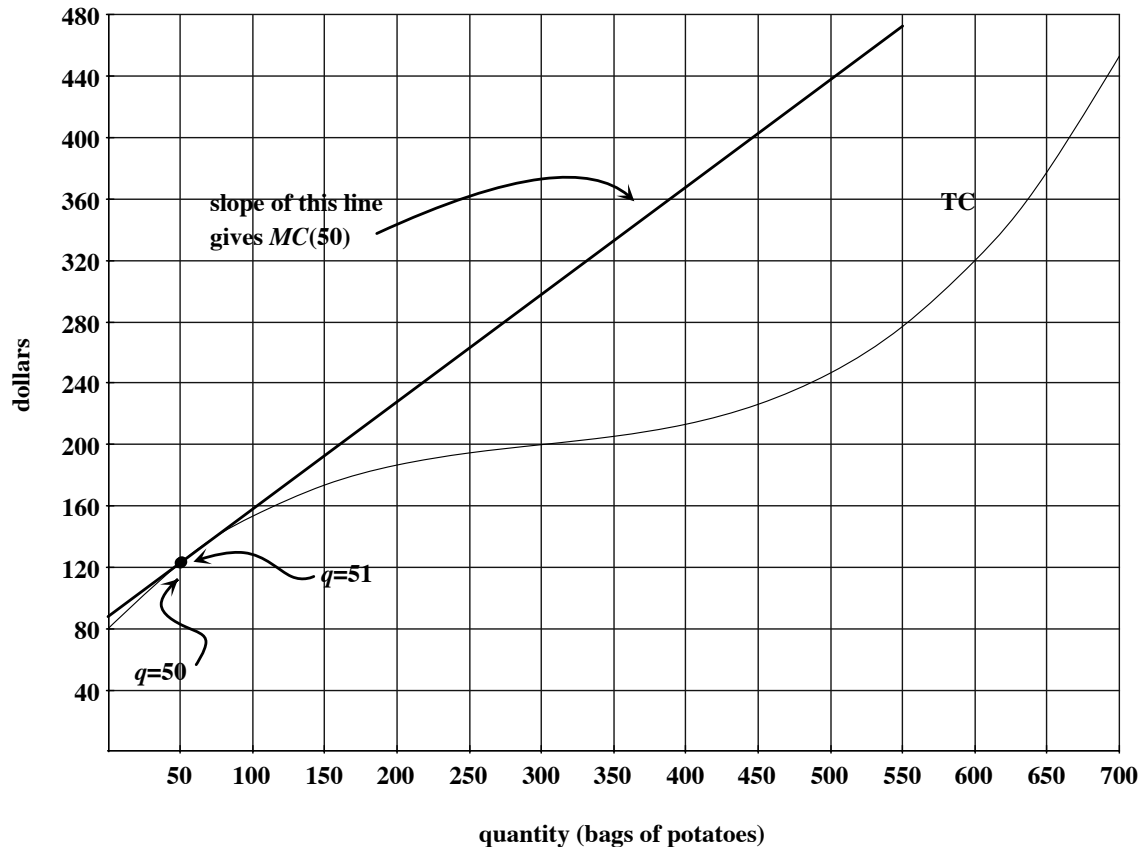
Recall that if the farmer produces 0 bags of potatoes, the only costs the farmer will pay are fixed costs. That is,  $TC(0) = FC$ . We can see the value of  $FC$  as the “ $y$ ”-intercept of the  $TC$  graph. In this case,  $FC = \$80$ .

We said before that marginal cost is the rate at which total cost changes. Specifically, we'll define the **marginal cost at  $q$  bags of potatoes** to be the *incremental rate of change in the total cost, in dollars per bag, as quantity changes from  $q$  to  $q + 1$  bags*. The incremental rate of change in total cost is given by the slope of a secant line through the total cost graph. So, for example, the marginal cost at 50 bags is the slope of the secant line through the total cost graph at  $q = 50$  and  $q = 51$ . We've magnified the  $TC$  graph at right so that we can clearly see the two points we need.



In order to compute the  $MC$ , however, we'll have to do our best to draw this secant line on the original  $TC$  graph and use techniques we learned previously:

- Draw the line through the  $TC$  graph at  $q = 50$  and  $q = 51$ . (With the scale of this graph, those two points look like they're practically on top of each other.)
- Extend the line as far as possible.
- Choose two points on the line, preferably with nice coordinates, that are far apart.
- Use those points to compute the slope.

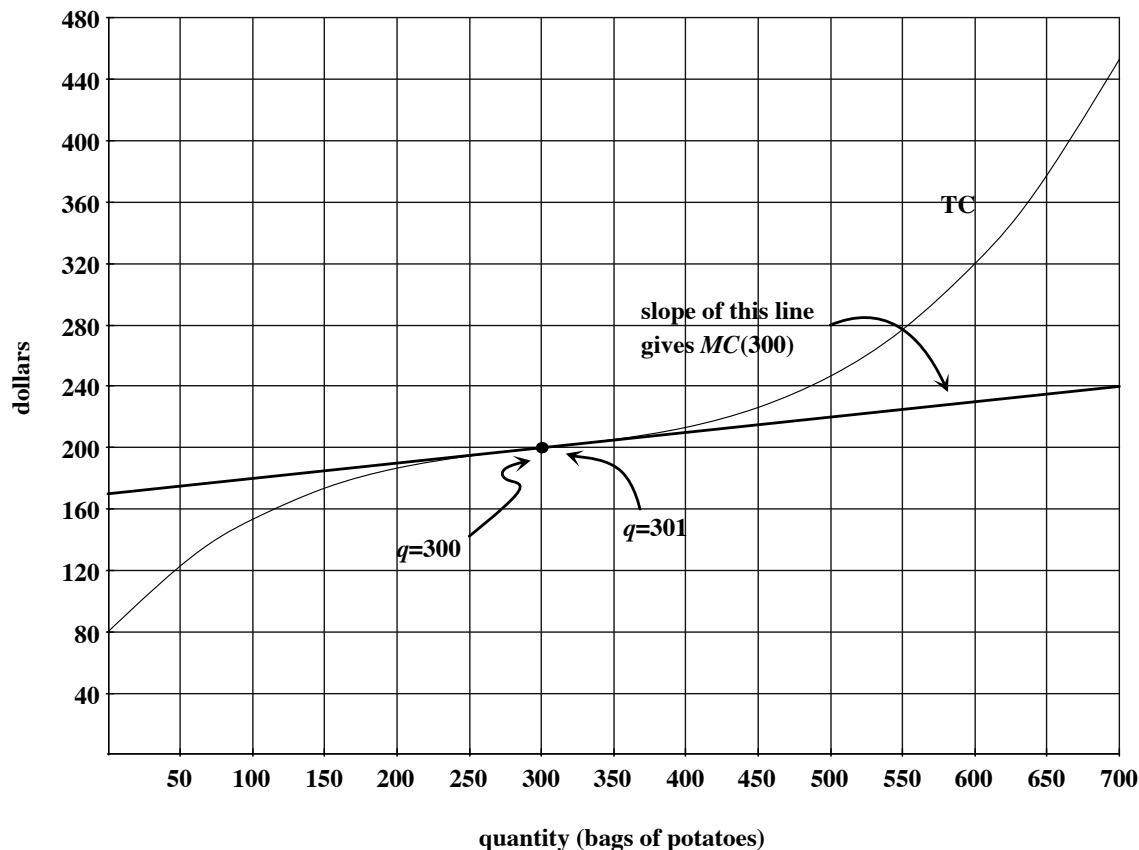


We'll approximate the coordinates of two points on this line:  $(0, 85)$  and  $(500, 440)$ . This gives

$$MC(50) \approx \frac{440 - 85}{500 - 0} = 0.71.$$

Notice that a *rise* on this graph is measured in dollars and a *run* on this graph is in bags of potatoes. This means that a slope of a line on this graph (rise/run) is in dollars per bag. So, once the farmer has produced 50 bags of potatoes,  $TC$  is increasing at a rate of \$0.71 per bag. You may think of it this way: to get from 50 to 51 bags, you'll need to spend an additional \$0.71. That is, the 51<sup>st</sup> bag of potatoes costs \$0.71 to produce.

Unlike with a linear  $TC$  graph, the rate at which  $TC$  changes varies from quantity to quantity. If we compute the marginal cost at 300 bags of potatoes, we will get a different rate. The marginal cost at 300 bags is the incremental rate of change in  $TC$  from  $q = 300$  to  $q = 301$ . This is given by the slope of the secant line through  $TC$  at  $q = 300$  and  $q = 301$ . We draw this line, choose the best points that we can on the line, and compute its slope.



Two points on this line have approximate coordinates:  $(300, 200)$  and  $(700, 240)$ . This gives

$$MC(300) \approx \frac{240 - 200}{700 - 300} = 0.1.$$

Once the farmer has produced 300 bags of potatoes,  $TC$  is increasing at a rate of 0.1 dollars per bag. So, the 301<sup>st</sup> bag of potatoes costs \$0.10 to produce.

## 7.2 Marginal Cost in Functional Notation

Regardless of the tricks we use to compute marginal cost, we go back to its definition and its graphical interpretation to translate into functional notation. With quantity  $q$  measured in bags of potatoes and total cost measured in dollars, we have:

$$\begin{aligned} MC(50) &= \text{incremental rate of change in } TC \text{ from } 50 \text{ to } 51 \\ &= \text{slope of secant line through } TC \text{ at } 50 \text{ and } 51 \\ &= \frac{TC(51) - TC(50)}{51 - 50} = \frac{TC(51) - TC(50)}{1} \frac{\text{dollars}}{\text{bag}}. \end{aligned}$$

Similarly,

$$MC(300) = \frac{TC(301) - TC(300)}{301 - 300} = \frac{TC(301) - TC(300)}{1} \frac{\text{dollars}}{\text{bag}}.$$

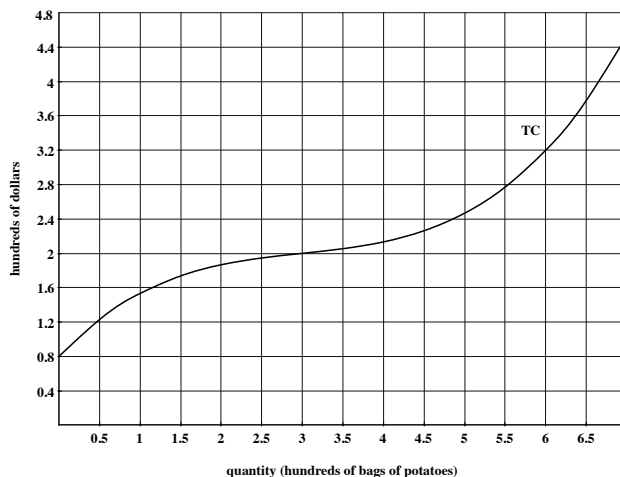
And in general,

$$MC(q) = \frac{TC(q+1) - TC(q)}{1} \frac{\text{dollars}}{\text{bag}}.$$

### 7.3 A Note About Units

The definition, graphical interpretation, and functional notation translation of marginal cost actually depend on the units you use to measure your production level and total cost. For example, if we alter the graph of total cost for our farmer's potato farm so that quantity is measured in **hundreds of bags** and total cost is in **hundreds of dollars**, then we'll have to make some adjustments to our idea of marginal cost.

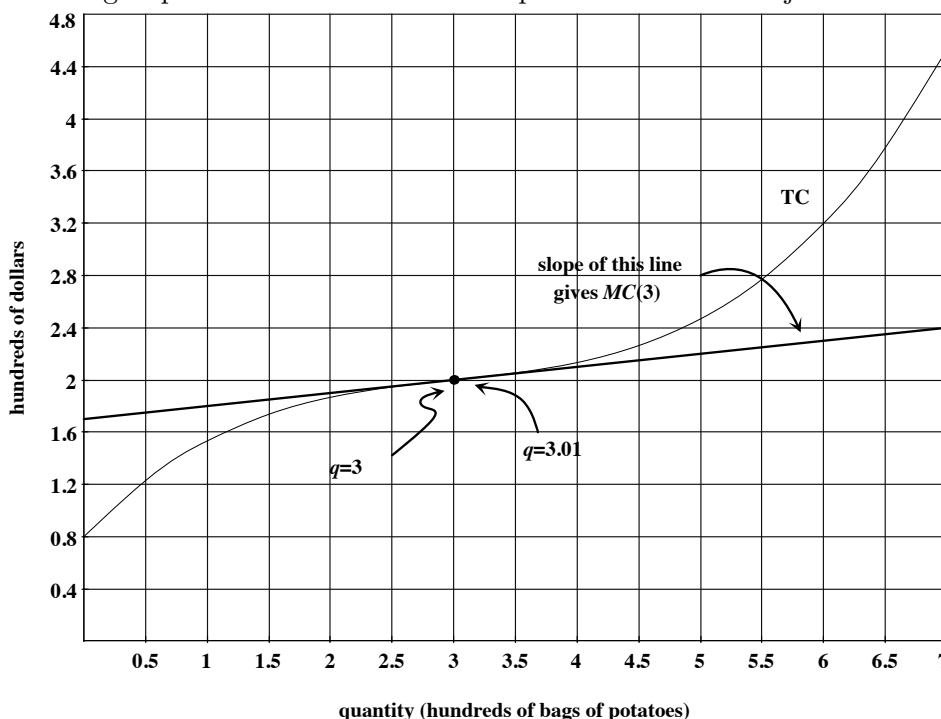
Marginal cost is the rate of change in total cost if quantity is increased by a **single bag** of potatoes. If the farmer has produced 3 hundred bags of potatoes, adding one more bag of potatoes means increasing quantity by 0.01 hundred bags (1 bag is  $\frac{1}{100}$  of 100 bags). With these units, marginal cost at 3 hundred bags is defined to be the incremental rate of change in total cost from  $q = 3$  to  $q = 3.01$  hundred bags. Graphically, this is the slope of the secant line through the  $TC$  graph at  $q = 3$  and  $q = 3.01$ . In functional notation, marginal cost at 3 hundred bags is



$$MC(3) = \frac{TC(3.01) - TC(3)}{3.01 - 3} \frac{\text{hundred dollars}}{\text{hundred bags}} = \frac{TC(3.01) - TC(3)}{0.01} \frac{\text{dollars}}{\text{bag}}.$$

Note that this slope calculation still gives marginal cost in units of dollars per bag—the *hundreds* cancel out.

To compute  $MC(3)$ , we'd draw the secant line through the  $TC$  graph at  $q = 3$  and  $q = 3.01$ , select two good points on that line, and compute its slope. You should check that the 301<sup>st</sup> bag of potatoes still costs \$0.10 to produce after this adjustment.



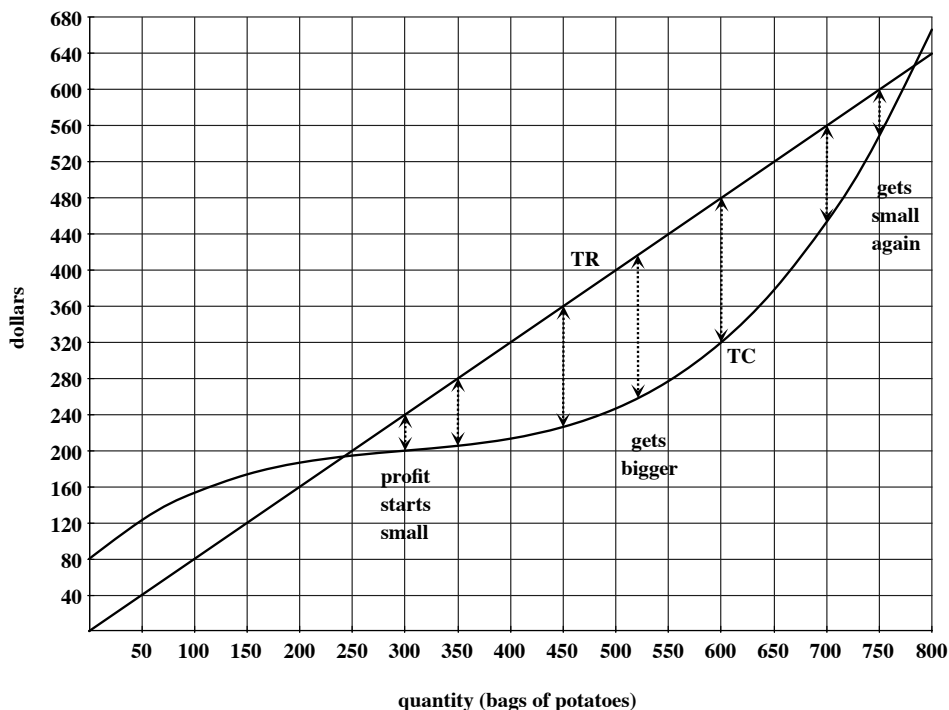
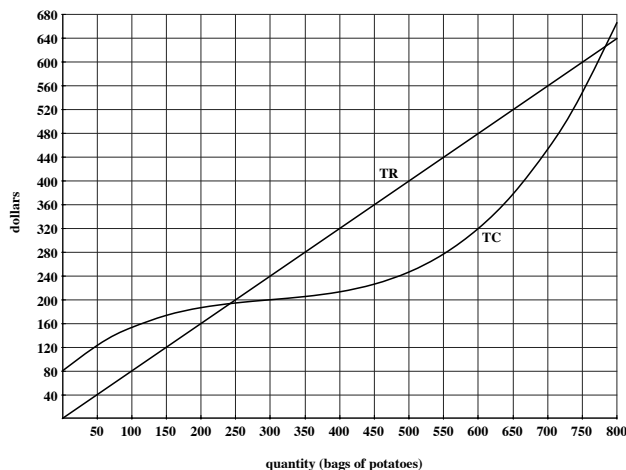
## 7.4 Marginal Revenue and Profit

We define **marginal revenue** to be the incremental rate of change of total revenue as quantity increases by one unit. Everything we discussed previously about marginal cost applies in a similar way to marginal revenue: in particular, if  $TR$  is linear, then  $MR$  is its slope; otherwise,  $MR$  will be the slope of the appropriate secant line through  $TR$ .

Let's go back to the original graph of total cost with quantity measured in bags of potatoes and add a linear total revenue graph. Suppose that the farmer sells each bag of potatoes for \$0.80. Then the graph of total revenue will be a diagonal line with slope 0.80.

Again, marginal revenue is the rate at which  $TR$  changes. Since every bag of potatoes sells for \$0.80,  $MR(q) = 0.80$  for every quantity  $q$ . In this section, we'll discuss what  $MR$  and  $MC$  can tell us about the behavior of profit.

Recall that profit is given by  $P(q) = TR(q) - TC(q)$  and that, when the  $TR$  graph is above the  $TC$  graph, we see profit as the vertical distance between  $TR$  and  $TC$ . On this graph,  $TR$  is above  $TC$  (i.e., profit is positive) from  $q \approx 240$  to  $q \approx 780$ . In between those two quantities, the vertical gap between  $TR$  and  $TC$  increases, reaches a maximum, and then decreases.



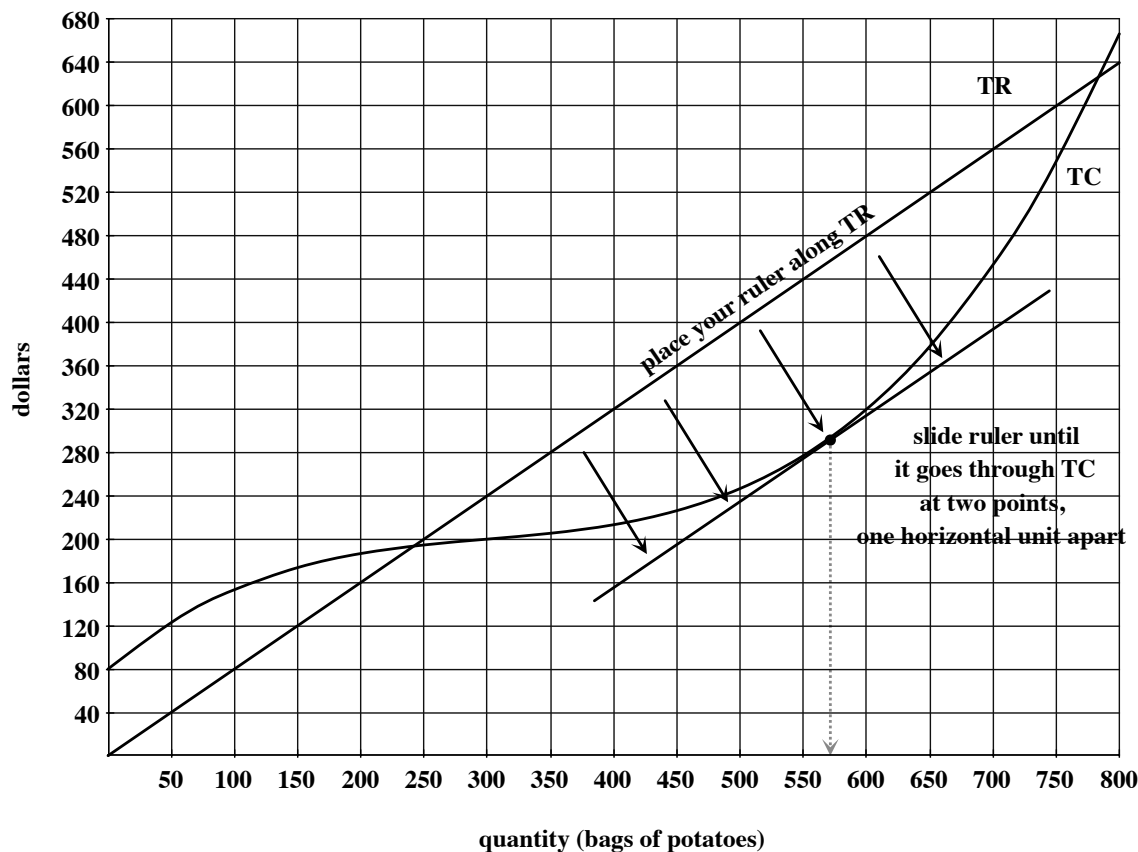
The relationship between marginal revenue and marginal cost will help us determine which quantity will yield the largest profit. We think of marginal revenue as the additional money the farmer will receive from selling one more bag of potatoes and marginal cost as the additional money the farmer must spend in order to produce one more bag. As long as marginal revenue is greater than marginal cost, producing and selling that extra bag will

cause profit to increase.

For example, we know that increasing production from 300 bags to 301 bags will increase the farmer's total revenue by \$0.80 (since at every quantity,  $MR(q) = 0.80$ ). We also know that this change in production will increase total cost by \$0.10 (since we computed  $MC(300) = 0.10$ ). At  $q = 300$  bags, marginal revenue exceeds marginal cost, which means that profit will go up if we sell that 301<sup>st</sup> bag. As another example, use the secant line method introduced earlier in this section to check that  $MC(700) \approx 1.7$ . This means that, if production increases from 700 to 701 bags, total cost will rise by \$1.70 while total revenue will only increase by \$0.80. This will cause profit to decrease if the farmer produces and sells the 701<sup>st</sup> bag of potatoes.

In general, profit is increasing when  $MR > MC$  and profit is decreasing when  $MR < MC$ . This means that profit must be largest at the transition: when  $MR = MC$ .

In order to find the quantity that maximizes profit, we must find the quantity at which  $MR(q) = MC(q)$ . Since  $TR$  is a line,  $MR(q)$  is always equal to 0.80, the slope of the  $TR$  graph. So we must find the quantity at which  $MC(q) = 0.80$ . We need to find two points on the  $TC$  graph, one horizontal unit apart, so that the secant line through those two points has slope 0.80. We use the graph of  $TR$  as a reference line and slide our ruler down:



It looks as though the secant line through the  $TC$  graph at  $q \approx 570$  and  $571$  bags is parallel to the  $TR$  graph. That is, marginal revenue and marginal cost are equal at  $q \approx 570$ . Since this is where profit changes from increasing to decreasing, profit is maximized at  $q \approx 570$ :

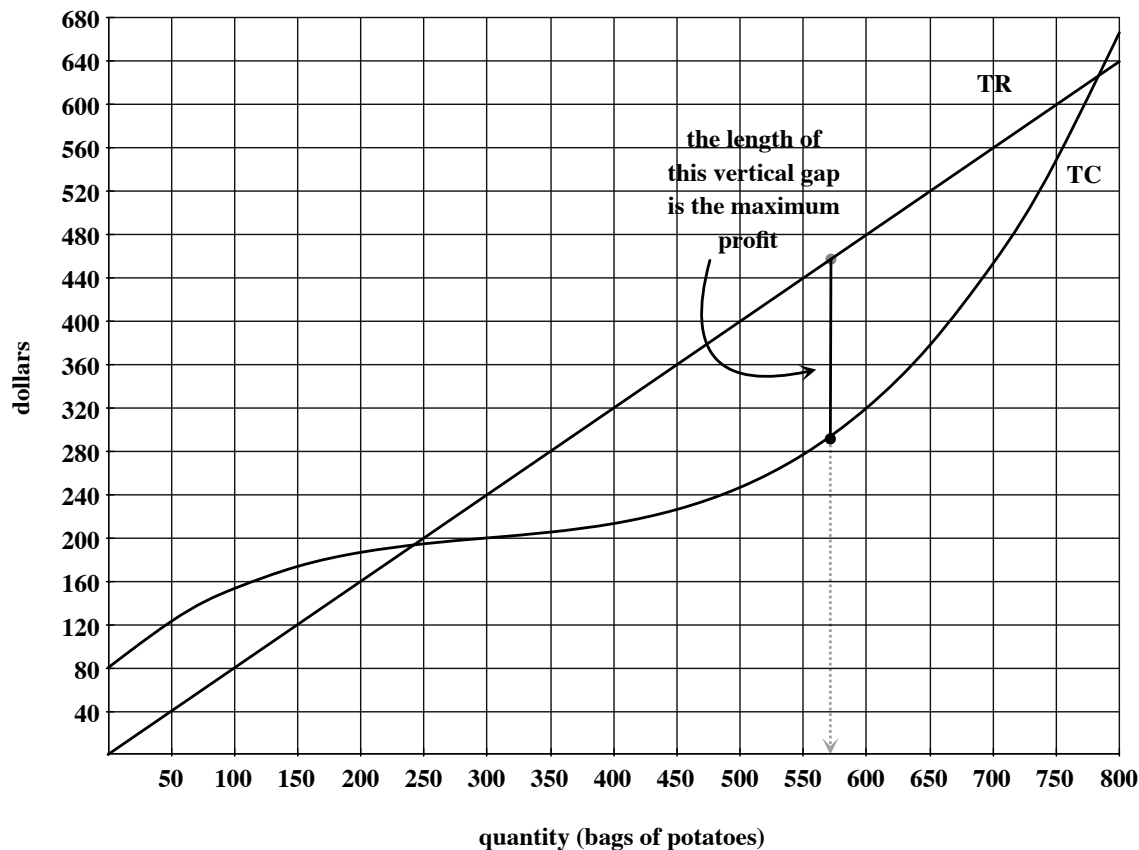
$MR$ bigger than $MC$	$MR = MC$	$MR$ smaller than $MC$
profit's increasing	profit's maximized	profit's decreasing
from $q \approx 240$ to $q \approx 570$	at $q \approx 570$	from $q \approx 570$ to $q \approx 770$

(We only need to consider quantities that give a *positive* profit. That's why we only look



at values of  $q$  between 240 and 770.)

Finally, we know that the farmer will maximize profit by producing and selling approximately 570 bags of potatoes. The length of the vertical gap between the  $TR$  and  $TC$  graphs at  $q \approx 570$  is about \$160:



To summarize:

- Marginal cost ( $MC$ ) is the incremental rate of change of total cost ( $TC$ ), measured in dollars per unit. This means:
  - If  $TC$  is linear,  $MC$  is its slope.
  - If quantity  $q$  is measured in single units,  $MC(q)$  is the slope of the secant line through  $TC$  at  $q$  and  $q + 1$ , which translates into functional notation as

$$MC(q) = \frac{TC(q+1) - TC(q)}{1}.$$

- Appropriate adjustments to this definition and formula must be made if quantity is measured in hundreds or thousands of units.
- Marginal revenue ( $MR$ ) is the incremental rate of change of total revenue ( $TR$ ), measured in dollars per unit. The same notes apply to  $MR$  as listed above for  $MC$ .
- Profit is maximized at a quantity  $q$  with the property that  $MR(q) = MC(q)$ .

## 8 Analysis of Cost

### 8.1 Average Cost and the Breakeven Price

We've already introduced total cost, variable cost, fixed cost, and marginal cost. In this section, we'll discuss two more kinds of cost and learn how to analyze the different cost functions to determine a good course of action when faced with forces of the market. Let's go back to the graph of total cost for the potato farmer from the last section.

We define **average cost** ( $AC$ ) at a quantity  $q$  by the formula

$$AC(q) = \frac{TC(q)}{q}.$$

This gives the average amount of money the farmer spends *per bag* to produce  $q$  bags of potatoes. For example,

$$AC(300) = \frac{TC(300)}{300} = \frac{\$200}{300 \text{ bags}} \approx \$0.67 \text{ per bag}.$$

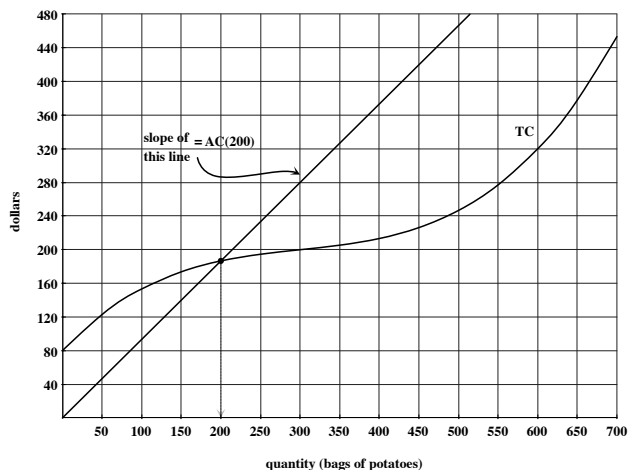
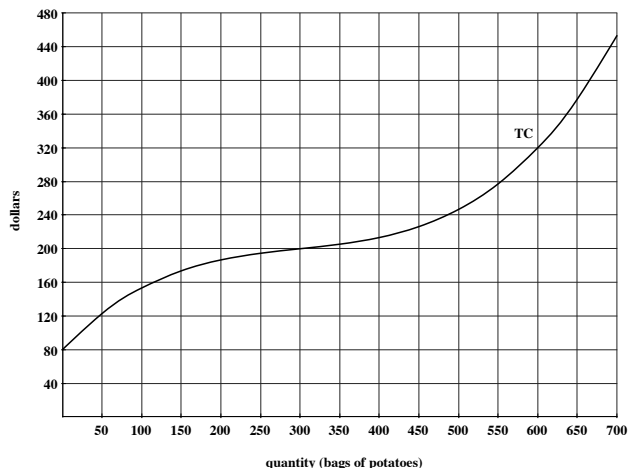
At a production level of 300 bags of potatoes, the farmer spends an average of \$0.67 on each bag of potatoes. (How is this different from  $MC(300)$ ?)

The translation of  $\frac{TC(300)}{300}$  into graphical language is the slope of the diagonal line through the  $TC$  graph at  $q = 300$ . In general, the graphical interpretation of  $AC(q)$  is the slope of the diagonal line through the  $TC$  graph at  $q$ . We can use the graphical interpretation to eliminate some of the error in approximating. For example, to compute  $AC(200)$ , draw the diagonal line through  $TC$  at  $q = 200$ , extending it as far as possible. Look for two points with nice coordinates and use those coordinates to compute the slope. In this example, we choose  $(0, 0)$  and  $(300, 280)$  as our two points. Then

$$AC(200) \approx \frac{280 - 0}{300 - 0} \approx 0.93.$$

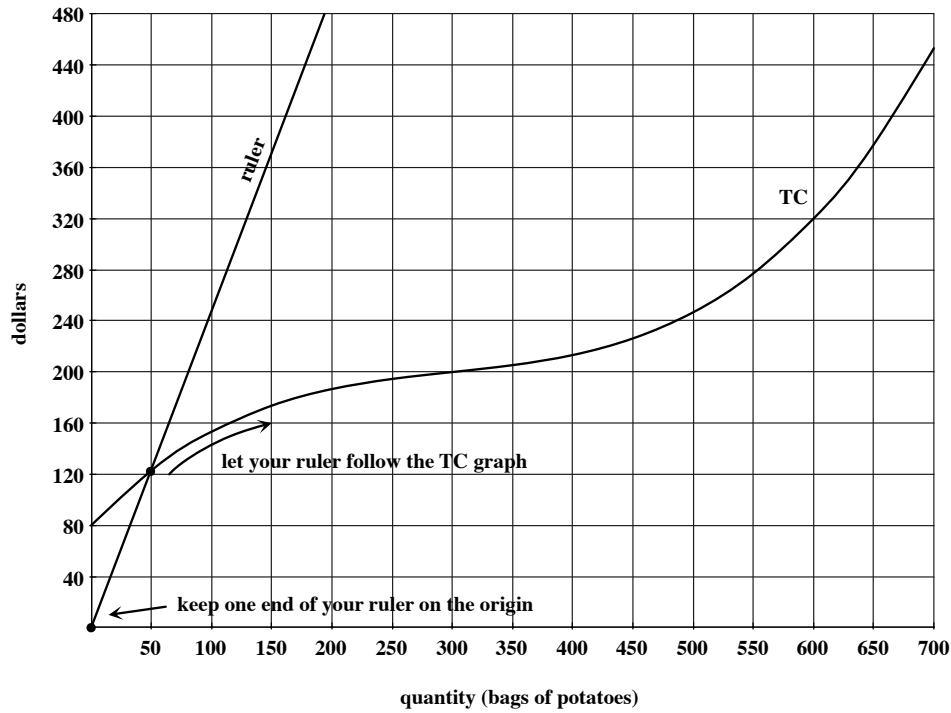
At a production level of 200 bags of potatoes, the farmer is spending an average of \$0.93 per bag.

Does it make sense that average cost at  $q = 200$  bags is higher than the average cost at  $q = 300$  bags? For relatively small production levels, it does make sense that average cost goes down as quantity goes up because the fixed costs get averaged over a larger quantity. It's reasonable to wonder, however, if average cost is *always* decreasing. The graphical

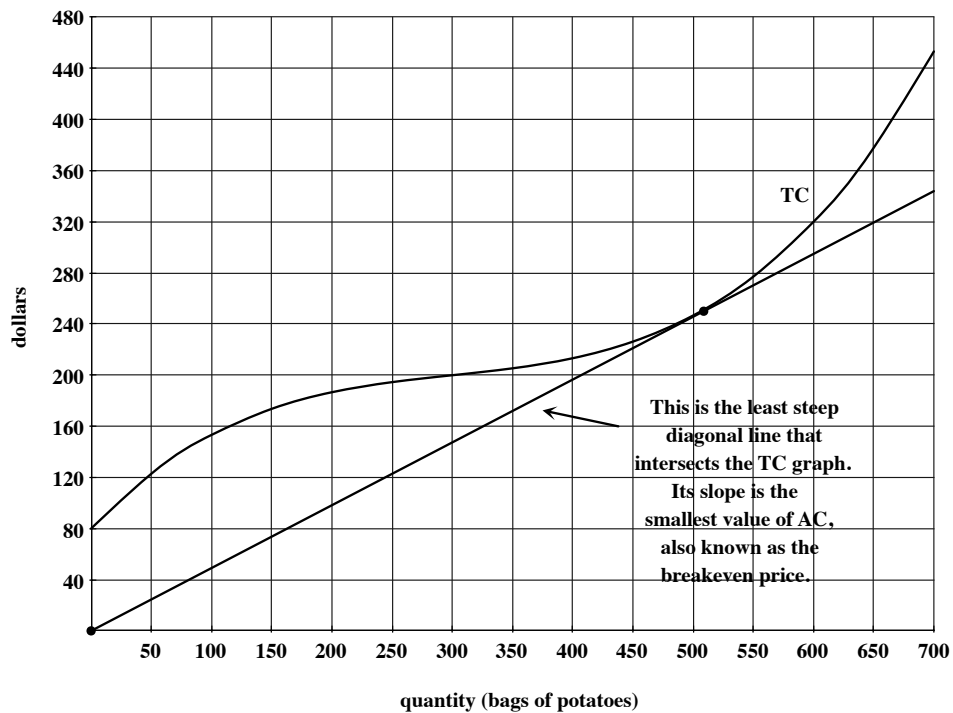


interpretation of average cost will help us determine the behavior of  $AC$  as production level increases.

Place your ruler on the graph so that it goes through the origin and another point on the  $TC$  graph at a fairly small value of  $q$ . Pivot your ruler so that it always goes through the origin and, at the same time, follows the curve of the  $TC$  graph.



Watch what happens to the diagonal lines as you pivot your ruler: they should get less steep for a while but then, in order to follow the curve, they will start to get steeper. That is, as you pivot your ruler, the slopes will decrease for a while, hit a minimum, and then start to increase again. The slope of the least steep diagonal line through the  $TC$  graph is the smallest value of average cost, sometimes called **the breakeven price**.



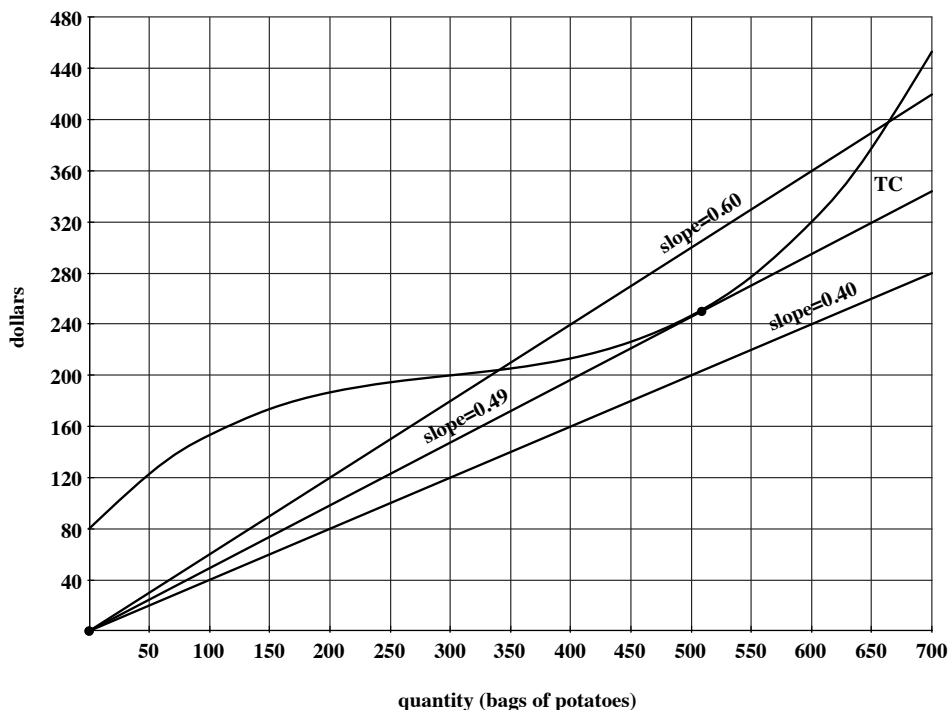
To compute the breakeven price, we draw the least steep diagonal line that intersects the

$TC$  graph, choose two points on that line, and compute its slope. Since this is a diagonal line, one point to choose is definitely  $(0, 0)$ . Another good one appears to be  $(650, 320)$ . Then the lowest value of  $AC$ , the breakeven price, is:

$$\frac{320 - 0}{650 - 0} \approx 0.49.$$

On this graph, a rise is measured in dollars and a run in bags of potatoes, so this slope is measured in dollars per bag. That is, the breakeven price is \$0.49 per bag.

To see why this is called the breakeven price, imagine that the market is such that the farmer must sell potatoes at a price of only \$0.40 per bag. Then the graph of total revenue will be a diagonal line with slope 0.40. This graph of  $TR$  will never intersect the graph of  $TC$ . This means that the graph of  $TR$  will always be below the graph of  $TC$  and profit will always be negative. If the market price for potatoes is less than the breakeven price, the farmer will be forced to take a loss at every possible production level. On the other hand, if the farmer can sell potatoes for \$0.60 per bag, then the graph of  $TR$  will intersect the graph of  $TC$  and there will be some quantities at which profit is positive.



The breakeven price is an important thing for the farmer to know:

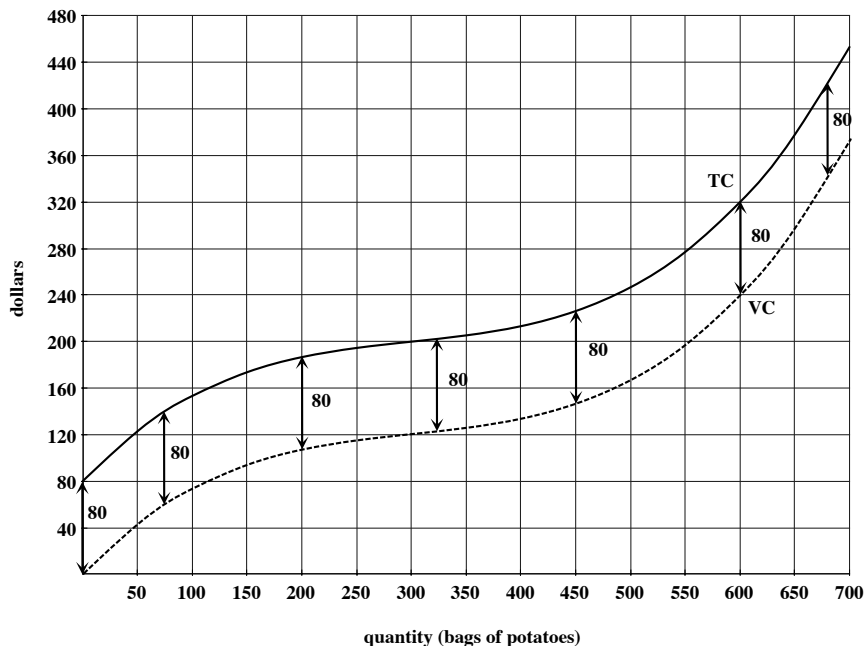
- If the market price for potatoes is **less than** the breakeven price, then the farmer **cannot** make a profit.
- If the market price for potatoes is **greater than** the breakeven price, then there are some levels of production at which the farmer **can** make a profit.

## 8.2 Variable Cost, Average Variable Cost, and the Shutdown Price

Recall that total cost is the sum of fixed cost and variable cost:  $TC(q) = VC(q) + FC$  for every quantity  $q$ . The potato farmer's fixed cost is \$80. So, in this example,  $TC(q) = VC(q) + 80$ . We can rearrange this equation as follows:

$$TC(q) - VC(q) = 80.$$

The graphical interpretation of this statement is: at every value of  $q$ , the vertical distance between the  $TC$  and  $VC$  graphs is 80. The graph below shows total cost and variable cost for producing potatoes.



In general,

- the graphs of  $TC$  and  $VC$  have the exact same shape;
- the graph of  $VC$  goes through the origin; and
- the vertical gap between  $TC$  and  $VC$  is always equal to the fixed cost.

We define **average variable cost** ( $AVC$ ) at a quantity  $q$  by the formula

$$AVC(q) = \frac{VC(q)}{q}.$$

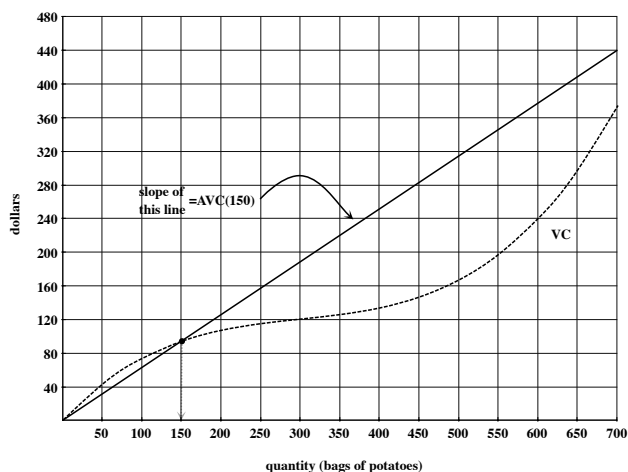
This gives the average amount of money the farmer spends per bag to produce  $q$  bags of potatoes if we disregard all fixed costs. For example,

$$AVC(300) = \frac{VC(300)}{300} = \frac{\$120}{300 \text{ bags}} = \$0.40 \text{ per bag.}$$

At a production level of 300 bags of potatoes, without including fixed costs, the farmer is spending an average of \$0.40 per bag.

Once again, we need a graphical interpretation of  $AVC(q)$ . We know that  $\frac{VC(q)}{q}$  is the slope of the diagonal line through the  $VC$  graph at  $q$ . So, for example, to compute  $AVC(150)$ , we draw the diagonal line through the  $VC$  graph at  $q = 150$ , extending it as far as possible. We find two points on this line:  $(0, 0)$  and  $(700, 440)$  look good. Then  $AVC(150)$  is the slope of the line that goes through these two points:

$$AVC(150) = \frac{440 - 0}{700 - 0} \approx 0.63.$$



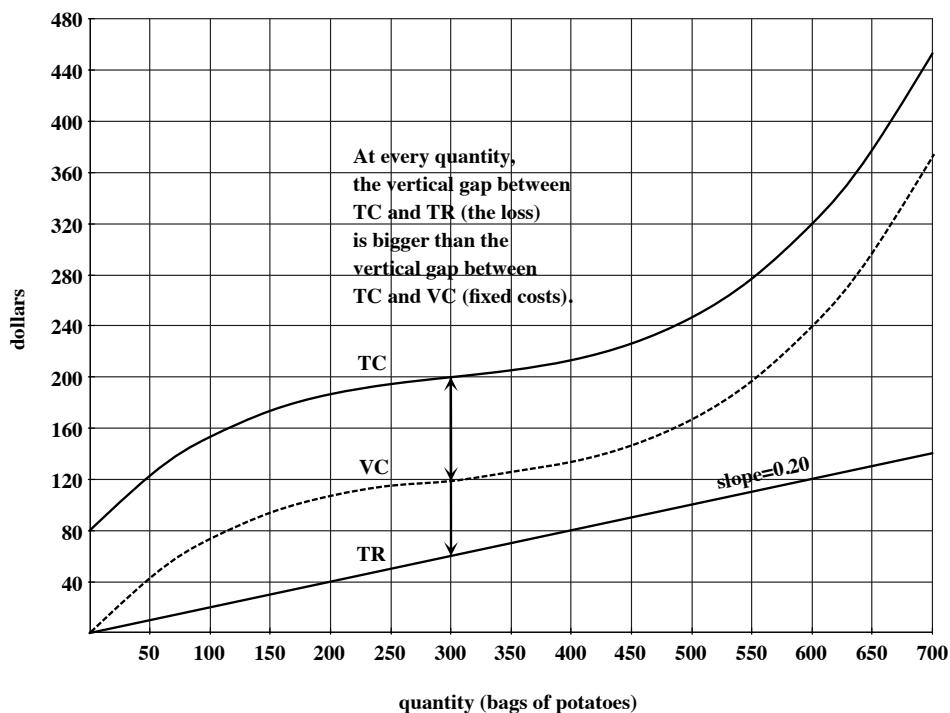
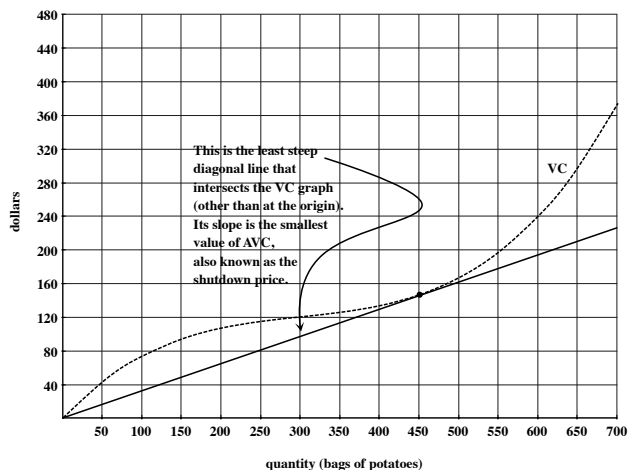
Again, this is measured in units of dollars per bag. (Why?)

Just like average cost, the values of average variable cost decrease for a while, hit a minimum, and then start to increase. (Use the “pivoting ruler” method you used before to see this. Keep one point on your ruler locked to the origin and let another point on the ruler follow the path of the  $VC$  graph from left to right.) The slope of the least steep diagonal line that intersects the variable cost graph is called **the shutdown price**.

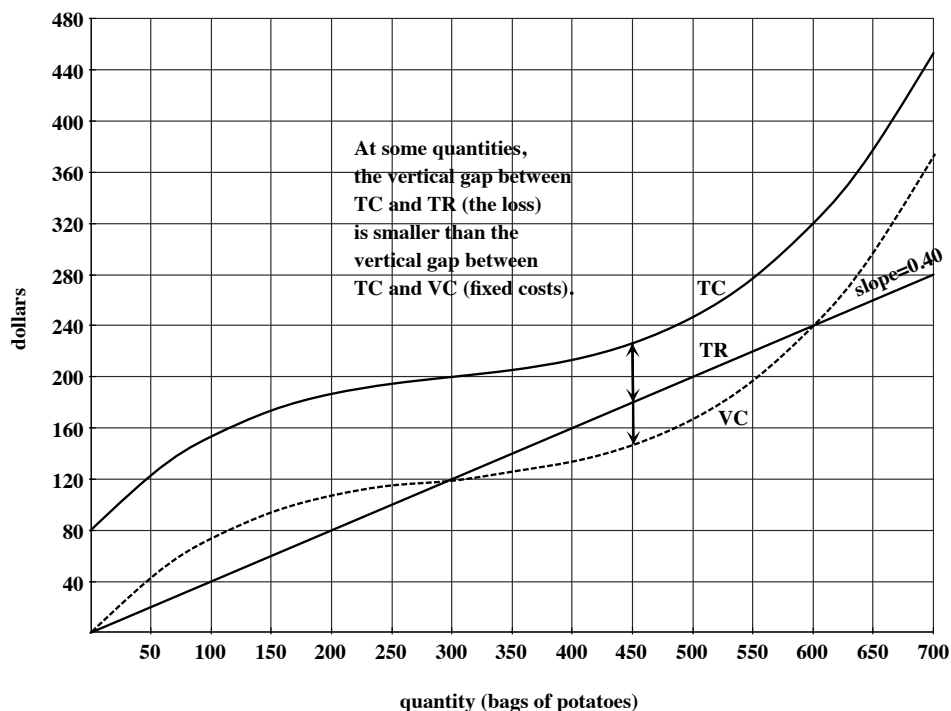
To compute the shutdown price, draw the least steep diagonal line that intersects the  $VC$  graph (at a point other than the origin). Choose two points on that line and compute its slope. You should verify that, for this potato farmer, the shutdown price is approximately \$0.32 per bag.

To understand the significance of the shutdown price, let’s look at two different possibilities.

**Market price is less than shutdown price:** Suppose the market price for potatoes is only \$0.20 per bag. Then the graph of  $TR$  is a diagonal line with slope 0.20. Since this is less than the shutdown price, the graph of  $TR$  is below the graph of  $VC$ , which in turn is below the graph of  $TC$ . So, the farmer cannot make a profit at any level of production. Moreover, at every quantity, the farmer’s loss (the vertical distance between  $TC$  and  $TR$ ) is larger than the fixed cost (the vertical distance between  $TC$  and  $VC$ ). The farmer must pay fixed costs no matter what. In this case, the farmer’s best strategy is to pay out the fixed costs and shut down.



**Market price is between the shutdown price and the breakeven price:** Suppose the market price for potatoes is \$0.40 per bag. This is larger than the shutdown price (\$0.32 per bag) and smaller than the breakeven price (\$0.49 per bag). The graph of  $TR$  will be a diagonal line with slope 0.40. Since \$0.40 is smaller than the breakeven price, we know the farmer cannot make a profit. However, since the farmer must pay the fixed costs no matter what, it turns out that it is in the farmer's best interest not to shut down in this case. Since the market price is greater than the shutdown price, the  $TR$  graph is above the  $VC$  graph for a range of quantities. At any of these quantities, the farmer's loss (the vertical distance between  $TC$  and  $TR$ ) is less than the fixed costs (the vertical distance between  $TC$  and  $VC$ ). Even though a positive profit is impossible, the farmer can lose less than the fixed costs by producing one of these quantities.



- If market price is less than shutdown price, you will minimize your losses by paying your fixed costs and shutting down.
- If market price is between the shutdown price and the breakeven price, you will not make a profit, but there will be production levels that will allow you to recoup some of your fixed costs.
- If market price is greater than the breakeven price, then there will be some production levels that will yield a (positive) profit.

### 8.3 Average Revenue

Very briefly, we define **average revenue**,  $AR$ , by the formula  $AR(q) = \frac{TR(q)}{q}$ . This gives the amount of money we receive on average for each unit sold. We often think of the average revenue as the price charged per unit sold. Its graphical interpretation is the slope of the diagonal line through the  $TR$  graph at  $q$ . Frequently,  $TR$  is linear. If this is the case, then for every quantity,  $AR(q)$  is equal to the slope of the  $TR$  graph, which is also the value of  $MR(q)$ . That is, when  $TR$  is linear,  $AR(q) = MR(q)$  for every quantity  $q$ .

## 8.4 Summary

Suppose you produce and sell Things. The following table summarizes the terms we've learned so far relating to revenue and cost. Assume you are given a graph of total cost  $TC(q)$  and total revenue  $TR(q)$  for producing and selling  $q$  Things.

Term	Definition	Related equations and formulas	Graphical Interpretation
total cost $TC(q)$	the total amount you spend to produce $q$ Things	$TC(q) = VC(q) + FC$	—
variable cost $VC(q)$	the money you spend to produce $q$ Things without including fixed costs	$VC(q) = TC(q) - FC$	the graph of $VC$ has the same shape as $TC$ and goes through the origin
fixed cost $FC$	the money you must spend even if you produce 0 Things; also known as <i>overhead</i>	$FC = TC(q) - VC(q)$ $FC = TC(0)$	the vertical distance between the $TC$ and $VC$ graphs OR the “ $y$ ”-intercept of the $TC$ graph
average cost $AC(q)$	total cost averaged over the number of Things produced	$AC(q) = \frac{TC(q)}{q}$	the slope of the diagonal line through the $TC$ graph at $q$
average variable cost $AVC(q)$	variable cost averaged over the number of Things produced	$AVC(q) = \frac{VC(q)}{q}$	the slope of the diagonal line through the $VC$ graph at $q$
breakeven price BEP	the smallest value of average cost	—	the slope of the least steep diagonal line that intersects the $TC$ graph
shutdown price SDP	the smallest value of average variable cost	—	the slope of the least steep diagonal line that intersects the $VC$ graph
marginal cost $MC(q)$ (see footnote)	the incremental rate of change in $TC$ from $q$ to $q + 1$ Things	$MC(q) = \frac{TC(q+1) - TC(q)}{1}$	the slope of the secant line through $TC$ (or $VC$ ) at $q$ and $q + 1$
total revenue $TR(q)$	the total amount you receive when you sell $q$ Things	—	—
average revenue $AR(q)$	total revenue averaged over the number of Things sold; also known as <i>price per Thing</i>	$AR(q) = \frac{TR(q)}{q}$	the slope of the diagonal line through the $TR$ graph at $q$
marginal revenue $MR(q)$ (see footnote)	the incremental rate of change in $TR$ from $q$ to $q + 1$ Things	$MR(q) = \frac{TR(q+1) - TR(q)}{1}$	the slope of the secant line through the $TR$ graph at $q$ and $q + 1$
profit $P(q)$	the money you are left with after subtracting total cost from total revenue	$P(q) = TR(q) - TC(q)$	the vertical distance between $TR$ and $TC$ (when $TR > TC$ )

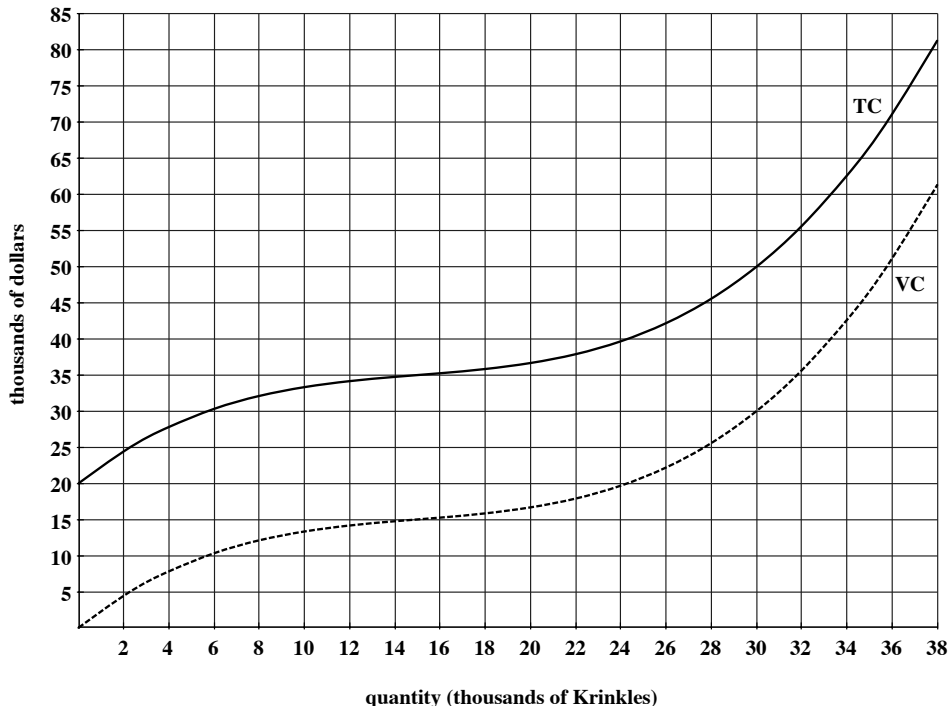
NOTE: If  $q$  is measured in hundreds or thousands of Things, the definitions, formulas, and graphical interpretations of marginal revenue and marginal cost must be adjusted appropriately.



## 9 More Analysis of Cost

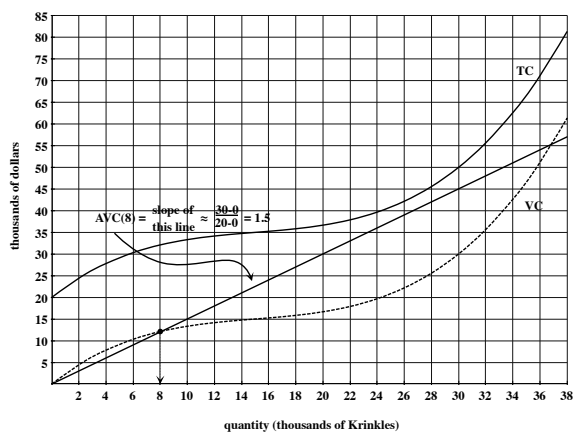
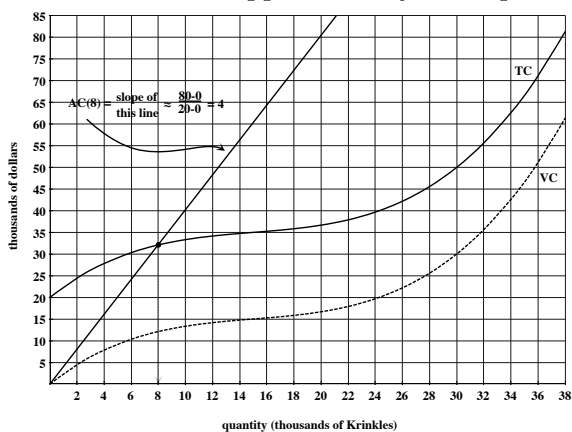
### 9.1 The Graphs of $AC$ , $AVC$ , and $MC$

Suppose you sell Krinkles. Here are the graphs of total cost and variable cost. Notice the units:  $q$  is measured in thousands of Krinkles and  $TC$  and  $VC$  are measured in thousands of dollars.

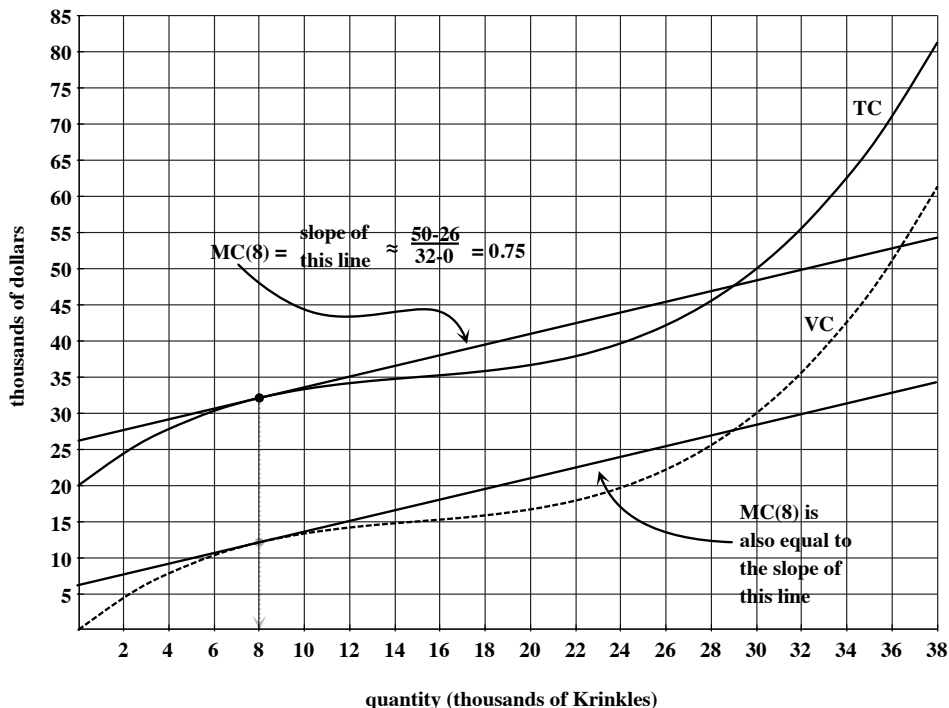


From these two graphs, we can generate the graphs of average cost, average variable cost, and marginal cost. This is tedious to do by hand—you'd generally use a spreadsheet to do work like this—but in order to help you see where these graphs come from, use graphical interpretations to compute  $AC$ ,  $AVC$ , and  $MC$  at  $q = 8$  thousand Krinkles.

To compute  $AC(8)$ , draw the diagonal line through  $TC$  at  $q = 8$ , select two points on that line (I picked  $(0, 0)$  and  $(20, 80)$ ), and compute its slope: 4. For  $AVC(8)$ , draw the diagonal line through  $VC$  at  $q = 8$ , select two points on that line (I picked  $(0, 0)$  and  $(20, 30)$ ), and compute its slope: 1.5. For both of these, a *rise* is measured in thousands of dollars and a *run* is in thousands of Krinkles. When we compute a slope on this graph, the *thousands* will cancel and we'll have units of dollars per Krinkle. When we produce 8 thousand Krinkles, our average cost is approximately \$4 per Krinkle and our average variable cost is approximately \$1.50 per Krinkle.



Marginal cost is the incremental rate of change of  $TC$  when we increase production by a single Krinkle. Since we're measuring quantity in thousands of Krinkles, marginal cost at  $q = 8$  thousand Krinkles is the incremental rate of change of  $TC$  from 8 to 8.001 thousand Krinkles. To compute  $MC(8)$ , draw the secant line through  $TC$  at  $q = 8$  and  $q = 8.001$ , choose two points on the line (I chose  $(0, 26)$  and  $(32, 50)$ ), and compute the slope: 0.75. Again, a slope on this graph will have units of dollars per Krinkle. When we produce 8 thousand Krinkles, our marginal cost is approximately \$0.75 per Krinkle (this is the cost to produce the 8,001<sup>st</sup> Krinkle).



(ASIDE: Notice here that, since  $TC$  and  $VC$  have exactly the same shape, the secant line through  $TC$  at  $q = 8$  and  $q = 8.001$  is *parallel* to the secant line through  $VC$  at the same two quantities. Parallel lines have the same slope. So, another graphical interpretation of  $MC(8)$  is the slope of the secant line through  $VC$  at  $q = 8$  and  $q = 8.001$ .)

We now have the following data:

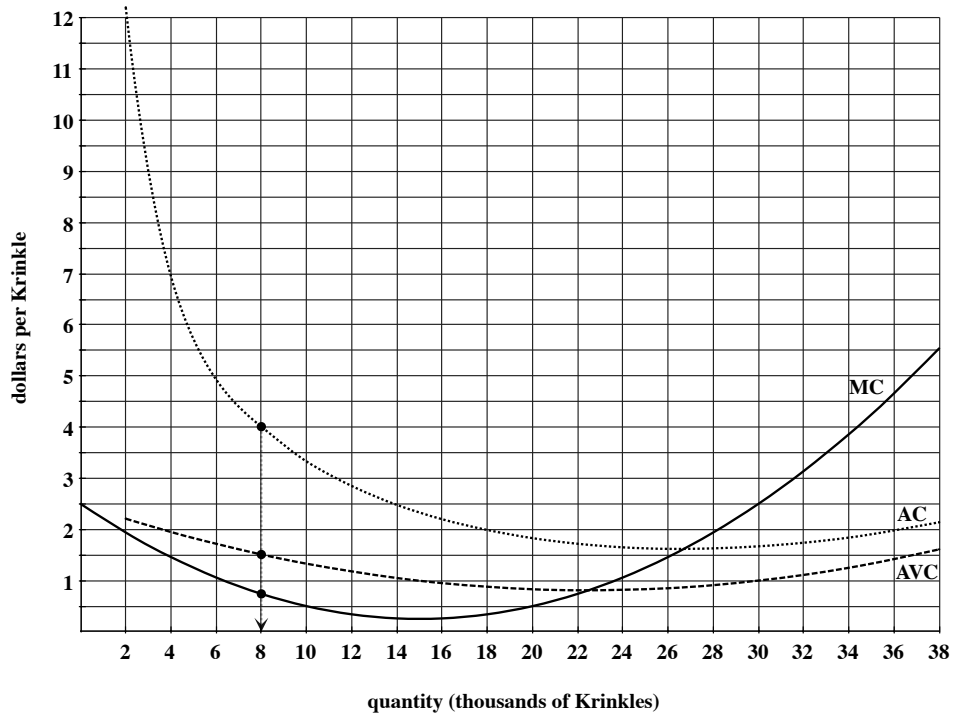
$q$	8
$AC(q)$	4
$AVC(q)$	1.5
$MC(q)$	0.75

Imagine repeating this process for many quantities. We'd get something like the following.

$q$	0	2	4	6	8	10	12	14	16	18	20
$AC(q)$	N/A	12.21	6.95	5.05	4	3.33	2.85	2.48	2.2	1.99	1.83
$AVC(q)$	N/A	2.21	1.95	1.72	1.5	1.33	1.18	1.05	0.95	0.88	0.83
$MC(q)$	2.5	1.94	1.46	1.06	0.75	0.5	0.34	0.26	0.26	0.34	0.5

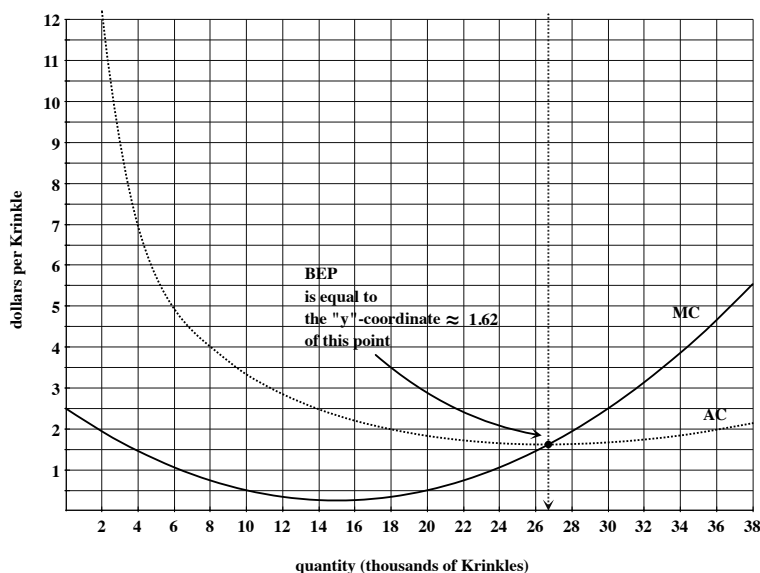
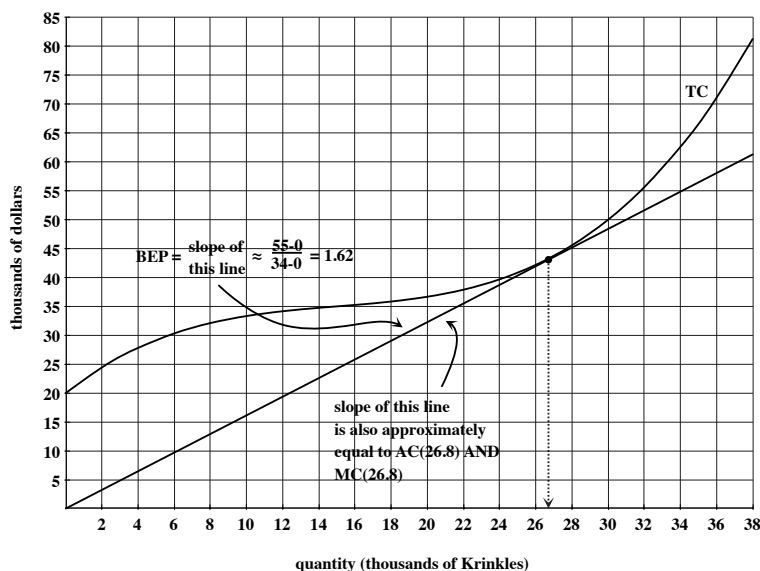
$q$	22	24	26	28	30	32	34	36	38
$AC(q)$	1.72	1.65	1.62	1.63	1.67	1.74	1.84	1.98	2.14
$AVC(q)$	0.81	0.82	0.85	0.91	1	1.11	1.25	1.42	1.61
$MC(q)$	0.74	1.06	1.46	1.94	2.5	3.14	3.86	4.66	5.54

Each value in this table is the slope of a line through the graph of  $TC$  or  $VC$ . Each value in this chart gives us a value measured in dollars per Krinkle. We can use these data to draw the graphs of  $AC$ ,  $AVC$ , and  $MC$ . We've highlighted the values we computed before for  $q = 8$ . Note that slopes on the graphs of  $TC$  and  $VC$  become “ $y$ ”-values on the graphs of  $AC$ ,  $AVC$ , and  $MC$ .



## 9.2 Using the Graphs of $MC$ and $AC$ to find Breakeven Price

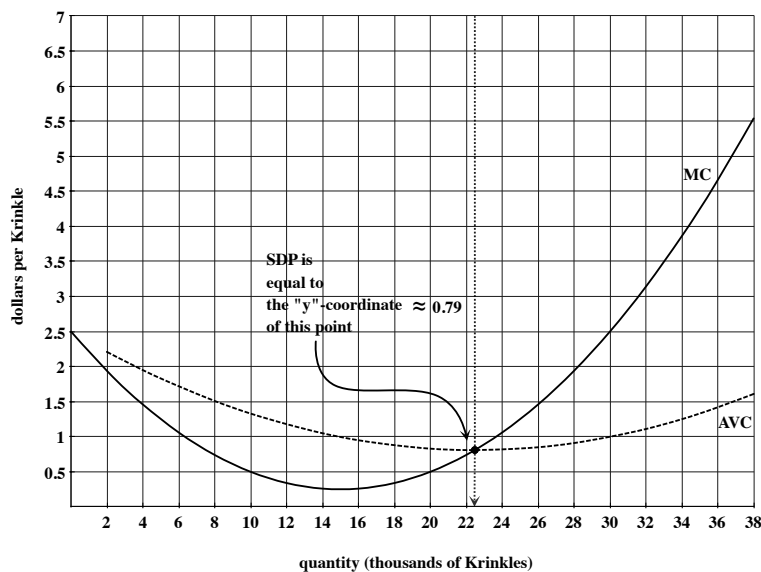
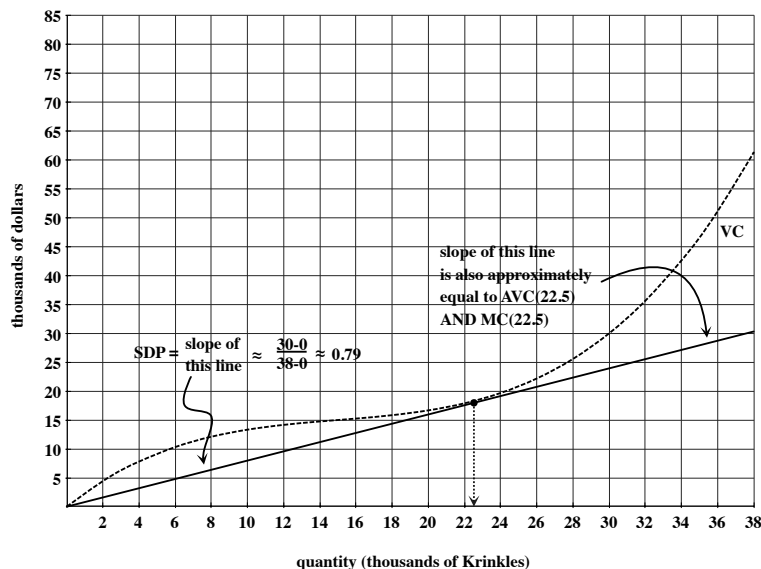
Recall that breakeven price ( $BEP$ ) is the smallest value of average cost. On the graph of  $TC$ , we find  $BEP$  by drawing the least steep diagonal line that intersects  $TC$  and computing its slope. In the figure below, we see that the  $BEP$  for selling Krinkles is the slope of the diagonal line that intersects the  $TC$  graph at  $q \approx 26.8$ . This means that the lowest value of  $AC$  occurs at  $q \approx 26.8$ . That is,  $BEP \approx AC(26.8)$ . But note that this slope could also be thought of as the  $MC(26.8)$  since this diagonal line is essentially the same as the secant line through  $TC$  at  $q \approx 26.8$  and  $q \approx 26.801$ . So,  $BEP$  is also approximately the value of  $MC(26.8)$ . When we look at the graphs of  $AC$  and  $MC$  that we created, we see that the two graphs intersect at  $q \approx 26.8$  and that  $AC$  is hitting its lowest point at  $q \approx 26.8$ . This means that the breakeven price is the “ $y$ ”-coordinate of the point of intersection of  $MC$  and  $AC$  OR the “ $y$ ”-coordinate of the lowest point on the  $AC$  graph.



Recall that, if market price is greater than the breakeven price, then there are quantities that will allow us to make a (positive) profit. So, for example, since breakeven price is approximately \$1.62, if market price is \$2.00 per Krinkle, we could make a profit; if market price is \$1.50 per Krinkle, we will be forced to take a loss. To determine whether we can recoup some of our fixed costs with a market price of \$1.50 or if we should simply shut down, we need our shutdown price.

### 9.3 Using the Graphs of $MC$ and $AVC$ to find Shutdown Price

Recall that shutdown price ( $SDP$ ) is the smallest value of average variable cost. On the graph of  $VC$ , we find  $SDP$  by drawing the least steep diagonal line that intersects  $VC$  and computing its slope. In the figure below, we see that the  $SDP$  for selling Krinkles is the slope of the diagonal line that intersects the  $VC$  graph at  $q \approx 22.5$ . This means that the lowest value of  $AVC$  occurs at  $q \approx 22.5$ . That is,  $SDP \approx AVC(22.5)$ . But note that this slope could also be thought of as the  $MC(22.5)$  since this diagonal line is essentially the same as the secant line through  $VC$  at  $q \approx 22.5$  and  $q \approx 22.501$ . So,  $SDP$  is also approximately the value of  $MC(22.5)$ . When we look at the graphs of  $AVC$  and  $MC$  that we created, we see that the two graphs intersect at  $q \approx 22.5$  and that  $AVC$  is hitting its lowest point at  $q \approx 22.5$ . This means that the shutdown price is the “ $y$ ”-coordinate of the point of intersection of  $MC$  and  $AVC$  OR the “ $y$ ”-coordinate of the lowest point on the  $AVC$  graph.



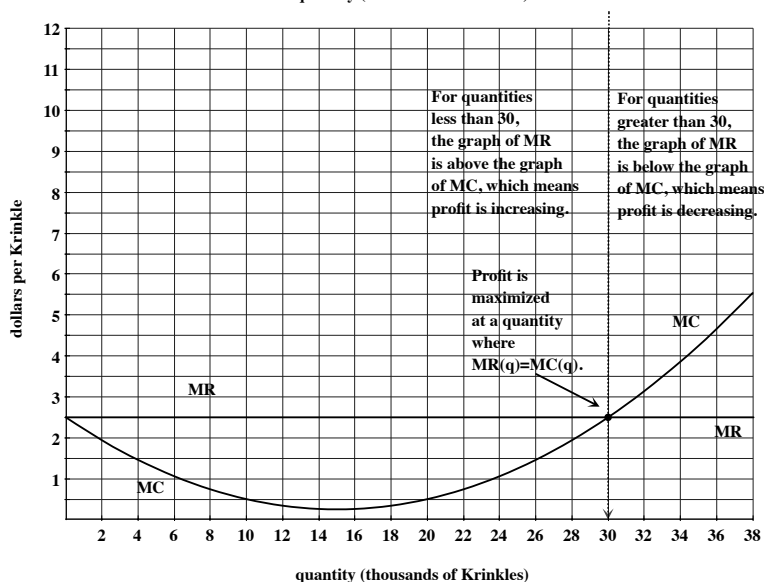
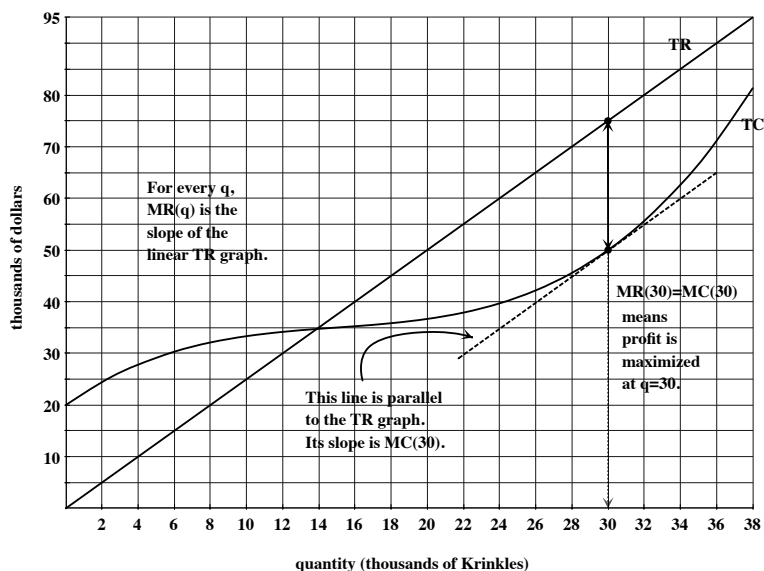
Recall that, if market price is less than the shutdown price, we minimize our losses by paying our fixed costs and shutting down. So, if Krinkles sell for only \$0.50 each, we should shut down. If market price is between the shutdown and breakeven prices, we can’t make a profit but we can recoup some of our fixed costs. So, if Krinkles sell for \$1.50 each, we should stay open to minimize our losses.

## 9.4 Generating and Using the Graph of $MR$

If total revenue is not linear, then we'd have to find the slopes of several secant lines through the  $TR$  graph to generate the graph of marginal revenue. Suppose in this case, however, that every Krinkle sells for \$2.50. Then the graph of  $TR$  is a diagonal line with slope 2.50. At every quantity,  $MR(q) = 2.50$  dollars per Krinkle. Then  $MR(0) = 2.50$ ,  $MR(2) = 2.50$ ,  $MR(4) = 2.50$ ,  $MR(6) = 2.50$ , etc. This means that the graph of marginal revenue will be a *horizontal line* on which every “ $y$ ”-coordinate is 2.50.

Given the graphs of **total revenue** and **total cost**, the maximum possible profit is the largest vertical gap between the  $TR$  and  $TC$  graphs (where  $TR$  is above  $TC$ ). This occurs at a value of  $q$  at which  $MR(q) = MC(q)$ —we can use the “sliding ruler” method from Section 7.4 to find this quantity.

Given the graphs of **marginal revenue** and **marginal cost**, the quantity that maximizes profit will be a value of  $q$  where the graphs intersect. If there is more than one such  $q$ , look for the one where the relative sizes of  $MR$  and  $MC$  change from  $MR > MC$  to  $MR < MC$ .



To summarize:

- Slopes of lines through  $TR$ ,  $TC$ , and  $VC$  become “ $y$ ”-values on the graphs of  $MR$ ,  $MC$ ,  $AC$ , and  $AVC$ .
- The breakeven price is the “ $y$ ”-coordinate of the lowest point on the  $AC$  graph.
- The breakeven price is the “ $y$ ”-coordinate of the point of intersection of the  $MC$  and  $AC$  graphs.
- The shutdown price is the “ $y$ ”-coordinate of the lowest point on the  $AVC$  graph.
- The shutdown price is the “ $y$ ”-coordinate of the point of intersection of the  $MC$  and  $AVC$  graphs.
- Profit is maximized at a quantity at which the  $MR$  and  $MC$  graphs intersect.