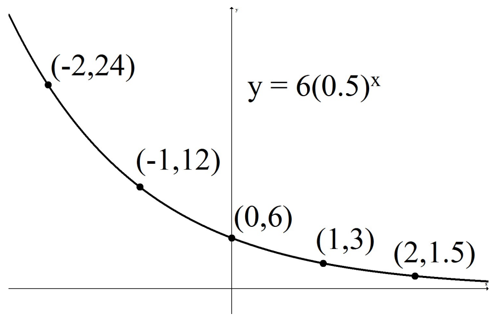
Sections 5.1-5.3 Review

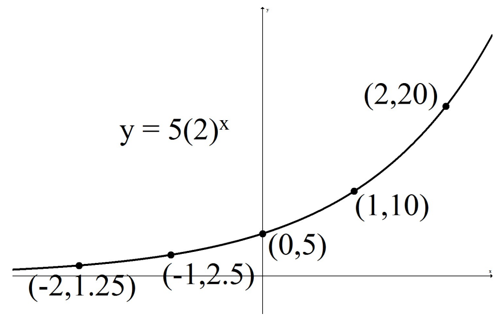
We introduce exponential and logarithmic functions in this section.

Basic Exponential Facts:

1. An exponential function is any function of the form $f(x) = A(b)^x$.

   - $f(0) = A =$ the y-intercept.
   - $b =$ the base.
   - If $0 < b < 1$, then we have exponential decay and if $b > 1$, then we have exponential growth. Here are examples:





2. The most commonly used based in calculus and applications is $e \approx 2.71828182$ (this is known as Euler's constant). We will see how this comes up when we start discussing compound interest. The graph of the function $y = e^x = (2.71828182)^x$ would look like an exponential growth graph that goes through the point $(0, 1)$.

3. In order to work with exponential functions, you first need to be comfortable with your exponent rules:

Rule	Example	Another Example
$b^0 = 1$	$2^0 = 1$	$e^0 = 1$
$b^x b^y = b^{x+y}$	$x^2 x^3 = x^5$	$e^x e^3 = e^{x+3}$
$(b^x)^y = b^{xy}$	$(x^2)^3 = x^6$	$e^{0.2x} = (e^{0.2})^x$
$b^{-x} = \frac{1}{b^x}$	$2^{-3} = \frac{1}{2^3}$	$e^{-x} = \frac{1}{e^x}$
$b^{1/2} = \sqrt{b}$	$9^{1/2} = \sqrt{9} = 3$	$\sqrt{e^x} = e^{x/2}$
$b^{1/n} = \sqrt[n]{b}$	$8^{1/3} = \sqrt[3]{8} = 2$	$\sqrt[5]{e^x} = e^{x/5}$

Basic Inverse Pairs Facts: The inverse of the exponential function is the logarithm function. More about logarithms on the next page. This a good place to stop and remind you of all your inverse pairs (these are the tools you use to solve equations).

Operation	Inverse	Example
$y = x + a$	$y - a = x$	$x + 3 = 10 \Rightarrow x = 10 - 3 = 7$
$y = x - a$	$y + a = x$	$x - 4 = 13 \Rightarrow x = 13 + 4 = 17$
$y = ax$	$\frac{y}{a} = x$	$4x = 12 \Rightarrow x = \frac{12}{4} = 3$
$y = \frac{x}{a}$	$ay = x$	$\frac{x}{5} = 10 \Rightarrow x = 5 \cdot 10 = 50$
$y = x^a$	$\sqrt[a]{y} = x$ (Include '$\pm$' for even roots)	$x^3 = 8 \Rightarrow x = \sqrt[3]{8} = 2$
$y = \sqrt[a]{x}$	$y^a = x$	$\sqrt[4]{x} = 3 \Rightarrow x = 3^4 = 81$
$y = b^x$	$\log_b y = x$ (You really only need $\ln(x) = \log_e x$)	$e^x = 3 \Rightarrow x = \ln(3) \approx 1.0986$
$y = \log_b x$	$b^y = x$	$\ln x = 3 \Rightarrow x = e^3$

For quadratic equations, we use the quadratic formula. But for all other equations that we can solve in Math 111 and Math 112, we try to get x by itself by using the inverses above (in the correct order).

For example, to solve $(2e^x)^3 + 7 = 15$, we

1. Use the inverse of $() + 7$ (which is $() - 7$) to get $(2e^x)^3 = 15 - 7 = 8$.
2. Use the inverse of $()^3$ (which is $\sqrt[3]{()}$) to get $2e^x = \sqrt[3]{8} = 2$
3. Use the inverse of $2 \cdot ()$ (which is $\frac{()}{2}$) to get $e^x = \frac{2}{2} = 1$.
4. Use the inverse of $e^{()}$ (which is $\ln()$) to get $x = \ln(1) = 0$.

Basic Logarithm Facts:

1. As mentioned on the last page the name we give to the inverses of exponential functions is logarithms.
$$y = b^x \quad \Longleftrightarrow \quad \log_b(y) = x.$$
2. Specifically, we will focus on
$$y = e^x \quad \Longleftrightarrow \quad \ln(y) = \log_e(y) = x.$$
We only need $\ln(x)$. In fact, $\log_b(x) = \frac{\ln(x)}{\ln(b)}$, so you can always convert to using natural logarithms. You don't even need to know that conversions, you just need to know how to use the natural logarithms rules which are listed below:

Rule	Example
$y = e^x \Longleftrightarrow x = \ln(y)$	$20 = e^x$ is the same as $\ln(20) = x$
$\ln(1) = 0$ and $\ln(e) = 1$	
$A = B \Rightarrow \ln(A) = \ln(B)$ (You can take the $\ln()$ of both sides)	$2^x = 10 \Rightarrow \ln(2^x) = \ln(10)$
$\ln(ab) = \ln(a) + \ln(b)$	$\ln(3 \cdot 5) = \ln(3) + \ln(5)$
$\ln(b^x) = x\ln(b)$ (Very important, we use this a lot!)	$\ln(2^x) = x\ln(2)$
$\ln(e^x) = x$ and $e^{(\ln(y))} = y$ (They are inverses!)	

3. Solving exponential equations: If we need to solve an equation that involves a variable in an exponent, then we need to do the following:
   (a) Isolate the exponential function: That is get $(b)^x$ by itself.
   (b) Take the natural logarithm of both sides and use the log rule: Remember that $\ln(b^x) = x\ln(b)$.
   (c) Divide by $\ln(b)$. Done!

   Here are two examples

   - Solve $10 + (3)^x = 90$.
     (a) Isolate the exponential: That is subtract 10 to get $(3)^x = 80$.
     (b) Take the $\ln()$ of both sides: So we get $\ln(3^x) = \ln(80)$ which becomes $x\ln(3) = \ln(80)$.
     (c) Divide by $\ln(3)$ to get: $x = \frac{\ln(80)}{\ln(3)} \approx \frac{4.382026635}{1.098612289} \approx 3.9887$.
   - Solve $30 - 2e^{0.1t} = 20$.
     (a) Isolate the exponential: Subtract 30 to get $-2e^{0.1t} = -10$ and divide by -2 to get $e^{0.1t} = 5$.
     (b) Take the $\ln()$ of both sides: So we get $\ln(e^{0.1t}) = \ln(5)$ which becomes $0.1t\ln(e) = \ln(5)$. Since $\ln(e) = 1$, we have $0.1t = \ln(5)$
     (c) Divide by 0.1 to get: $x = \frac{\ln(5)}{0.1} \approx \frac{1.609437912}{0.1} \approx 16.0944$.

Here is a random collection of equations to solve, try them out (solutions below and on the next page):

1. Solve $12 - 5e^{3t} = 3$
2. Solve $2x^3 - 4 = 10$
3. Solve $4\ln(2t + 5) = 10$.
4. Solve $\sqrt{x-3} + 7 = 12$.
5. Solve $200000 = 350000(1 - e^{-0.07t})$.
6. Solve $3 - \frac{5}{x} = 1$.
7. Solve $14400 = 1200(7)^{4r}$.
8. Solve $(2^{3x} - 1)^5 + 10 = 40$
9. Solve $x + \frac{2}{x} = 10$.
10. Solve $5 = \frac{60}{e^x+1}$.

Solutions to exercises:

1. Answer:

$12 - 5e^{3t}$	$= 3$	
$-5e^{3t}$	$= -9$	subtracted 12
e^{3t}	$= \frac{9}{5}$	divided by -5
$3t$	$= \ln\left(\frac{9}{5}\right)$	take ln() of both sides
t	$= \frac{\ln\left(\frac{9}{5}\right)}{3} \approx 0.1959$	divided by 3

2. Answer:

$2x^3 - 4$	$= 10$	
$2x^3$	$= 14$	added 4
x^3	$= 7$	divided by 2
x	$= (7)^(1/3) \approx 1.9129$	cube root

3. Answer:

$4\ln(2t+5)$	$= 10$	
$\ln(2t+5)$	$= \frac{10}{4} = 2.5$	divided by 4
$2t + 5$	$= e^{2.5}$	exponentiated both sides
$2t$	$= e^{2.5} - 5$	subtracted 5
t	$= \frac{e^{2.5}-5}{2} \approx 3.59125$	divided by 2

4. Answer:

$\sqrt{x-3} + 7$	$= 12$	
$\sqrt{x-3}$	$= 5$	subtract 7
$x - 3$	$= 25$	square both sides
x	$= 28$	add 3

5. Answer:

$350000(1-e^{-0.07t})$	$= 200000$	
$1-e^{-0.07t}$	$= \frac{200000}{350000} \approx 0.571428$	divided by 350000
$-e^{-0.07t}$	≈ -0.428571	subtracted 1
$e^{-0.07t}$	≈ 0.428571	divided by -1
$-0.07t$	$\approx \ln(0.428571) \approx -0.84729786$	take ln() of both sides
t	$\approx \frac{-0.84729786}{-0.07} \approx 12.1043$	divided by -0.07

6. Answer:

$3-\frac{5}{x}$	$= 1$	
$3x-5$	$= x$	multiply both sides by x
$3x$	$= x+5$	add 5
$2x$	$= 5$	subtract x
x	$= \frac{5}{2} = 2.5$	divide by 2

7. Answer:

$1200(7)^{4r}$	$= 14400$	
$(7)^{4r}$	$= \frac{14400}{1200} = 12$	divide by 1200
$\ln(7^{4r})$	$= \ln(12)$	taking ln() of both sides
$4t\ln(7)$	$= \ln(12)$	bringing down exponent
t	$= \frac{\ln(12)}{4\ln(7)} \approx 0.31925$	divide by $4\ln(7)$

8. Answer:

$(2^{3x}-1)^5+10$	$= 40$	
$(2^{3x}-1)^5$	$= 30$	subtract 10
$2^{3x}-1$	$= (30)^{1/5} \approx 1.9743505$	fifth root
2^{3x}	≈ 2.9743505	add 1
$\ln(2^{3x})$	$\approx \ln(2.9743505) \approx 1.09002569$	take ln() of both sides
$3x\ln(2)$	≈ 1.09002569	bring down exponent
x	$\approx \frac{1.09002569}{3\ln(2)} \approx 0.524192$	divide by $3\ln(2)$

9. Answer:

$x+\frac{2}{x}$	$= 10$	
x^2+2	$= 10x$	multiply by x (Quadratic! Make one side zero!)
$x^2-10x+2$	$= 0$	subtract $10x$
x	$= \frac{-(-10)\pm\sqrt{(-10)^2-4(1)(2)}}{2(1)}$	quad. form. $a=1$, $b=-10$, $c=2$
x	$\approx \frac{10\pm 9.5916630566}{2}$	simplifying

We get the two answer $x \approx \frac{10-9.5916630566}{2} \approx 0.204168$ or $x \approx \frac{10+9.5916630566}{2} \approx 9.7958315$.

10. Answer:

5	$= \frac{60}{e^x+1}$	
$5(e^x+1)$	$= 60$	multiply by (e^x+1)
e^x+1	$= \frac{60}{5} = 12$	divide by 5
e^x	$= 11$	subtract 1
x	$= \ln(11) \approx 2.397895$	take ln() of both sides