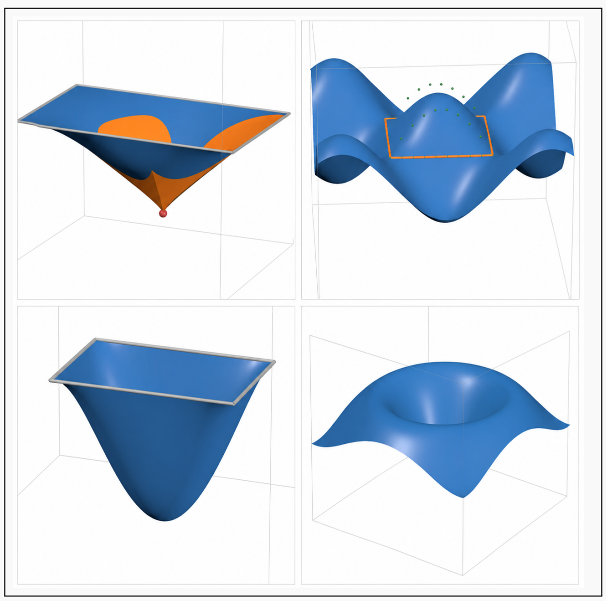
0.1 Aiden 2 - How Geometry Shapes Vibration

The Idea. What happens when a flat surface, like a drumhead, is pulled or hit and then released? This project explores the two-dimensional wave equation, which models how a surface vibrates over time. The main idea is that the shape of the membrane affects the types of waves that can appear.

The Story / Why we care. Waves appear in many real-life situations, including sound, water, earthquakes, and musical instruments. For this project, I focused on vibrating membranes. A membrane is like a flexible surface whose edge is held fixed. When it moves, the motion is not random. Instead, it can be broken into natural patterns called modes.

A major thing I learned is that geometry matters. A rectangular membrane and a circular membrane both follow the same wave equation, but their solutions look very different. Rectangles lead to sine functions, while circles lead to Bessel functions. This shows that the shape of an object changes the way it vibrates.

Visual.



Try this.

- How does the vibration change when the starting shape of the membrane changes?
- What differences do you notice between the rectangular membrane and the circular membrane?

Interactive / Video.

- Desmos links:
  - Rectangle, Circular, Fundamental

The Mathematics.

The two-dimensional wave equation is

$$u_{tt} = c^2(u_{xx} + u_{yy}).$$

Here, $u(x, y, t)$ represents the height of the membrane at point (x, y) and time t. The constant c is the wave speed. The left side, u_{tt}, describes how the surface accelerates in time. The right side, $u_{xx} + u_{yy}$, describes how curved the surface is in space.

For a fixed membrane, the boundary stays still:

$$u = 0 \quad \text{on the boundary.}$$

This means the edge of the membrane is always held down.

For a rectangular membrane with side lengths L_x and L_y, the natural vibration modes are

$$\phi_{mn}(x, y) = \sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right),$$

where m and n are positive integers. The number m controls the number of waves in the x-direction, and n controls the number of waves in the y-direction.

Each mode has frequency

$$\omega_{mn} = c\pi\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}.$$

This means changing the size or shape of the rectangle changes the vibration frequencies.

For a circular membrane, rectangular sine functions are no longer the most natural choice. Instead, we use polar coordinates (r, θ). The circular modes involve Bessel functions:

$$\phi_{mn}(r, \theta) = J_m\left(\frac{\alpha_{mn} r}{R}\right)\cos(m\theta - \varphi).$$

Here, R is the radius of the circle, J_m is a Bessel function, and α_{mn} is a zero of that Bessel function. The zeros are important because they make the edge of the circular membrane stay fixed.

How to use this.

- The wave equation can model vibrating surfaces such as drums or membranes.
- Circular membranes use Bessel functions because the boundary is round.

Questions to explore.

- How does changing the shape of the membrane change the sound?
- What happens if the boundary is not fixed?

Notes / Further reading.

- Project webpage: 2D Wave Equation: How Geometry Shapes Vibration