

Integration of rational functions is mostly a matter of algebraic manipulation. In this worksheet we shall work through some examples of the necessary techniques.

1a Consider the rational function $f(x) = \frac{2x^3 - 4x^2 - 5x + 3}{x^2 - 2x - 3}$. Use long division to get a quotient and a remainder, then write $f(x) = \text{quotient} + (\text{remainder}/\text{divisor})$.

$$f(x) = 2x + \frac{x + 3}{x^2 - 2x - 3}$$

1b Now consider the expression $\frac{x + 3}{x^2 - 2x - 3}$. Factor the denominator into two linear terms.

$$\frac{x + 3}{(x + 1)(x - 3)}$$

1c We wish to write $\frac{x + 3}{(x - 3)(x + 1)}$ as a sum $\frac{A}{x - 3} + \frac{B}{x + 1}$. Let's find A and B . Set the two expressions equal and clear denominators (that is, multiply through by $(x - 3)(x + 1)$ and cancel $(x - 3)$'s and $(x + 1)$'s as much as possible). Plug in $x = 3$ and solve for A . Use the same idea to find B . Check your work by adding the two fractions together.

$$x + 3 = A(x + 1) + B(x - 3)$$

$$\text{If } x = 3, \text{ we get } 6 = 4A \text{ so } A = \frac{3}{2}.$$

$$\text{If } x = -1, \text{ we get } 2 = -4B \text{ so } B = -\frac{1}{2}.$$

$$\text{Check that } \frac{3/2}{x - 3} + \frac{-1/2}{x + 1} = \frac{x + 3}{(x - 3)(x + 1)}.$$

1d Now use the results of Problems 1a, 1b, and 1c to compute $\int \frac{2x^3 - 4x^2 - 5x + 3}{x^2 - 2x - 3} dx$.

$$\text{Putting it all together, } f(x) = 2x + \frac{3/2}{x - 3} - \frac{1/2}{x + 1} \text{ so}$$

$$\int f(x) dx = \int 2x + \frac{3/2}{x - 3} - \frac{1/2}{x + 1} dx = x^2 + \frac{3}{2} \ln|x - 3| - \frac{1}{2} \ln|x + 1| + C.$$

1e Some of the terms in the answer to Problem 1d involve logarithms. Combine those terms into a single term of the form $\ln(\text{some function of } x)$.

$$\int f(x) dx = x^2 + \ln \sqrt{\left| \frac{(x - 3)^3}{x + 1} \right|} + C$$

2a Next, consider the function $f(x) = \frac{3x+1}{x(x+1)^2}$. The problem here is that one of the linear factors in the denominator is squared. Partial fraction theory says the best we can do is to get this one in the form $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. Let's find A , B and C .

First set the two expressions equal and clear denominators. Plug in $x = 0$ and solve for A . Plug in $x = -1$ and solve for C .

$$3x + 1 = A(x + 1)^2 + Bx(x + 1) + Cx$$

If $x = 0$, we get $1 = A$.

If $x = -1$, we get $-2 = -C$ so $C = 2$.

2b Now that you've found A and C , you can find B by plugging in any other convenient value for x . Do so.

$$\text{We have } 3x + 1 = (x + 1)^2 + Bx(x + 1) + 2x.$$

Plug in $x = 1$, say, to get $4 = 2^2 + 2B + 2$ so that $B = -1$.

2c Now compute $\int \frac{3x+1}{x(x+1)^2} dx$.

$$\text{We have } \frac{3x+1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} + \frac{2}{(x+1)^2}.$$

(It's a good idea to check that this is indeed true.)

$$\text{Then } \int \frac{3x+1}{x(x+1)^2} dx = \int \frac{1}{x} - \frac{1}{x+1} + \frac{2}{(x+1)^2} dx = \ln|x| - \ln|x+1| - \frac{2}{x+1} + C.$$

$$\text{This can also be written } \ln\left|\frac{x}{x+1}\right| - \frac{2}{x+1} + C.$$