SOLUTIONS:

1. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int x\tan^2 x\,dx$

$= \int x(\sec^2 x - 1)\,dx$ IBP: $u = x$ & $dv = (\sec^2 x - 1)\,dx$, $du = dx$ $v = \tan x - x$

$= x(\tan x - x) - \int \tan x - x\,dx$

$= x\tan x - x^2 - \ln|\sec x| + \frac{x^2}{2} + C$

$= \boxed{x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C}$

OR

$= \underbrace{\int x\sec^2 x\,dx}_{} - \int x\,dx$ IBP: $u = x$, $du = dx$ $dv = \sec^2 x\,dx$, $v = \tan x$

$= x\tan x - \int \tan x\,dx - \frac{x^2}{2}$

$= x\tan x - \ln|\sec x| - \frac{x^2}{2} + C$

(b) $\int_{-2}^{4} |x+1|\,dx$

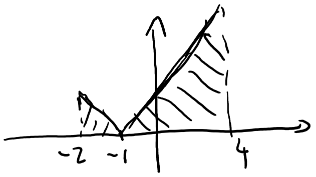
$= \int_{-2}^{-1}(-x-1)\,dx + \int_{-1}^{4}(x+1)\,dx$

$= \left(-\frac{x^2}{2} - x\right)\Big|_{-2}^{-1} + \left(\frac{x^2}{2} + x\right)\Big|_{-1}^{4}$

$= \left(-\frac{1}{2} + 1\right) - \left(\underset{0}{\cancel{-2+2}}\right) + (8+4) - \left(\frac{1}{2} - 1\right)$

$= \frac{1}{2} + 12 + \frac{1}{2} = \boxed{13}$.

OR $|x+1| = \begin{cases} x+1 & \text{for } x \geq -1 \\ -(x+1) & \text{for } x \leq -1 \end{cases}$



$A_1 + A_2 = \frac{1}{2}(1)(1) + \frac{1}{2}(5)(5)$

$= \frac{1}{2} + \frac{25}{2} = \boxed{13}$.

2. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int \frac{1}{1+x^{2/3}}dx$

$\underline{\text{Substitute } u = x^{1/3}} \Rightarrow x = u^3 \Rightarrow \underline{dx = 3u^2 du}$

$= \int \frac{3u^2}{1+u^2} du$

$\underline{\text{Long division}}$ $u^2+1 \overline{\big) 3u^2}$ with quotient 3; $-\underline{3u^2-3}$; $\cancel{\ } -3.$ $(\text{or } \underline{\text{algebra}})$

$= \int \left(3 - \frac{3}{1+u^2}\right) du$

$= 3u - 3\arctan(u) + C$

$= \boxed{3\sqrt[3]{x} - 3\arctan\sqrt[3]{x} + C}$

(b) $\int x\sqrt{5+4x-x^2}\,dx$

Complete the square:

$-(x^2-4x-5) = -[(x-2)^2-4-5] = 9-(x-2)^2$

$= \int x\sqrt{9-(x-2)^2}\,dx$

Trig sub: $x-2 = 3\sin\theta$
$dx = 3\cos\theta\, d\theta$

$= \int (3\sin\theta+2)\underbrace{\sqrt{9-9\sin^2\theta}\; 3\cos\theta}\, d\theta$

$= \int (3\sin\theta+2)\overbrace{9\cos^2\theta}\, d\theta$

$= \int 27\cos^2\theta\sin\theta\, d\theta + \int 18\cos^2\theta\, d\theta$

$= \int 27u^2(-1)\,du + \int 18\frac{1+\cos 2\theta}{2}\, d\theta$

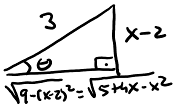
$= -9u^3 + 9\left[\theta + \frac{\sin(2\theta)}{2}\right] + C$

$= -9\cos^3\theta + 9\theta + \frac{9}{\cancel{2}}\cancel{2}\sin\theta\cos\theta + C$

$= -\cancel{9}\frac{(\sqrt{5+4x-x^2})^3}{\cancel{27}\,3} + 9\arcsin\left(\frac{x-2}{3}\right) + \cancel{9}\frac{(x-2)\sqrt{5+4x-x^2}}{\cancel{9}} + C$

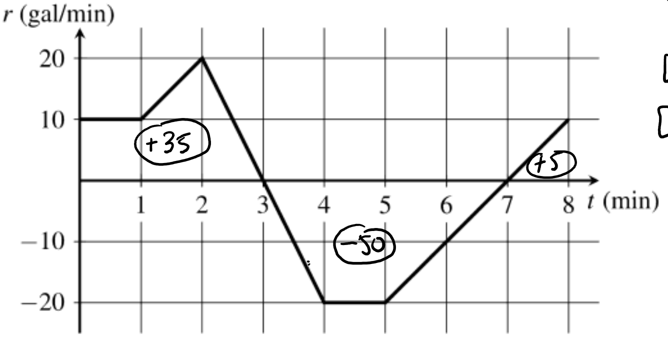
$= \boxed{-\frac{1}{3}\left(\sqrt{5+4x-x^2}\right)^3 + 9\arcsin\left(\frac{x-2}{3}\right) + (x-2)\sqrt{5+4x-x^2} + C}$

$\sin\theta = \frac{x-2}{3} = \frac{\text{opp}}{\text{hyp.}}$



$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{5+4x-x^2}}{3}$

3. (10 points) The following graph depicts a function $r(t)$ that gives the rate of flow of water into a tank, measured in gallons per minute. At time $t = 0$ minutes, the tank contains 100 gallons of water.



From graph/areas

$[0,3]: +35$ gal

$[3,7]: -50$ gal

$[7,8]: +5$ gal.

(a) What is the maximum amount of water in the tank at any time? Show work/justify.

$$\text{Max water} = 100 + \int_0^3 r(t)\,dt = 100 + \text{area under graph over } [0,3]$$

$$= 100 + (3.5)(10) = \boxed{135 \text{ gal.}}$$

(b) At which time(s) is the amount of water in the tank decreasing the fastest?

$r(t) = -20$ $\boxed{\text{from } t=4 \text{ to } t=5 \text{ minutes}}$

(c) At which time(s) does the tank have the least amount of water in it?

$\boxed{\text{at } t=7 \text{ minutes}}$

(d) What is the average rate of flow of water into the tank over the first 4 minutes? Show work.

$$r_{\text{ave}} = \frac{1}{4}\int_0^4 r(t)\,dt = \frac{1}{4}\left[+35-10\right] = \boxed{\frac{25}{4}\text{ gal/min}}.$$

(e) Let $f(x) = \int_1^{x^2} r(t)\,dt$. Find $f'(2)$, showing your steps.

$$f'(x) = r(x^2)\cdot(2x)$$

$$f'(2) = r(4)\cdot 4 = (-20)(4) = \boxed{-80}$$

4. (10 points) A high-speed bullet train travels between two consecutive stations that are 30 km apart. Find the time it takes the train to travel between these two stations, if the train starts at rest at the first station, accelerates at 10 km/min^2 until it reaches its maximum cruising speed of 3 km/min, drives at that speed for as long as possible, then decelerates at 5 km/min^2 in time to stop at the second station.

① While accelerating:

- start at rest: $v_1(0)=0$
  accelerate at $a_1(t)=10$ km/min^2 $\Big] \Rightarrow v_1(t)=10t$
- until it reaches 3 km/min: $10t_1=3 \Rightarrow t_1=0.3$ min
- displacement while accelerating: $d_1=\int_0^{0.3}10t\,dt=5t^2\Big|_0^{0.3}=0.45$ km

[So: the train takes $\boxed{t_1=0.3\text{ min}}$ and travels $\boxed{d_1=0.45\text{ km}}$ in order to accelerate from rest to max. cruising speed.

③ While decelerating (and taking t to mean time since it starts braking)

- start at $v_3(0)=3$ km/min
  decelerate at $a_3=-5$ km/min $\Big] \Rightarrow v_3(t)=-5t+3$
- until it comes to a stop: $-5t_3+3=0 \Rightarrow t_3=3/5=0.6$ min.
- displacement: $d_3=\int_0^{0.6}(-5t+3)\,dt=\left(-\frac{5}{2}t^2+3t\right)\Big|_0^{0.6}=0.9$ km.

[So: the train takes $\boxed{t_3=0.6\text{ min}}$ & $\boxed{d_3=0.9\text{ km}}$ to come to a full stop.

② At max cruising speed:

- The stations are 30 km apart, so the distance that the train travels at constant max. speed $v_2=3$ km/min is:

$$d_2=30\text{ km}-d_1-d_3=30\text{ km}-0.45\text{ km}-0.9\text{ km}$$

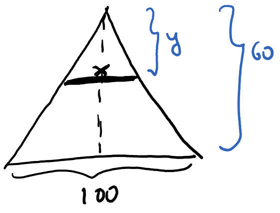
$$\boxed{d_2=28.65\text{ km}}.$$

- Which takes: $t_2=\frac{28.65\text{ km}}{3\text{ km/min}} \Rightarrow \boxed{t_2=9.55\text{ min}}$

Total time: $t_1+t_2+t_3=0.3+9.55+0.6=\boxed{10.45\text{ min}}$ ANSWER ←

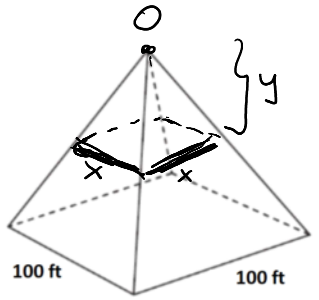
5. (10 points) A solid pyramid that is 60 feet tall with a square base that is 100 feet wide is to be built out of limestone. (A cubic foot of limestone weighs 175 pounds.)

(a) Set up (but DO NOT EVALUATE) an integral for the **total weight, in pounds**, of the limestone needed to build the pyramid.



$$\frac{y}{60} = \frac{x}{100}$$

$$\therefore x = \frac{100}{60} y = \frac{5}{3} y$$



$$\text{Volume} = \int_0^{60} A(y)\,dy = \int_0^{60} x^2\,dy$$

$$\text{Weight} = (175)(\text{volume}) = \begin{cases} \boxed{175 \int_0^{60} \left(\frac{25}{9} y^2\right) dy} & \text{if origin = at top of pyramid.} \\ \text{OR} & \\ \boxed{175 \int_0^{60} \frac{25}{9} (60-y)^2\, dy} & \text{if origin = at bottom of pyramid.} \end{cases}$$

(b) Set up (but DO NOT EVALUATE) an integral for the **work, in foot-pounds**, needed to raise the limestone to the required height to build the pyramid, assuming all limestone starts in a quarry that is 150 feet below the base of the pyramid.

With origin at the top of the pyramid

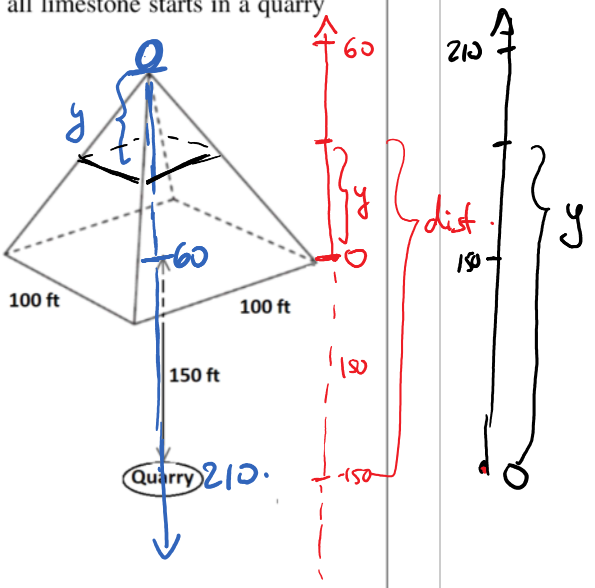
$$\boxed{W = \int_0^{60} 175 \left(\frac{25}{9} y^2\right) (210-y)\, dy}$$

OR, with origin at the base of the pyramid:

$$\boxed{W = \int_0^{60} 175 \left(\frac{25}{9} (60-y)^2\right) (150+y)\, dy}$$

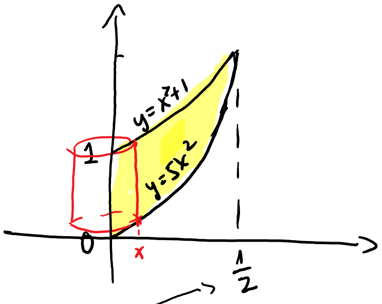
OR, with origin at the quarry:

$$\boxed{W = \int_{150}^{210} 175 \left(\frac{25}{9} (210-y)^2\right) (y)\, dy}$$



6. (10 points) The region in the first quadrant to the right of the y-axis, above the curve $y = 5x^2$, and below the curve $y = x^2 + 1$ is rotated around the y-axis to form a solid of revolution.

   (a) Set up a definite integral for the volume of this solid using the method of cylindrical shells, and evaluate the integral to compute the volume.



Intersection point(s): $5x^2 = x^2+1$

$\Leftrightarrow 4x^2 = 1 \Leftrightarrow x^2 = \frac{1}{4} \Leftrightarrow x = \pm\frac{1}{2}$

$$V = \int_0^{1/2} 2\pi(\text{radius})(\text{height})\,dx$$

$$= \boxed{\int_0^{1/2} 2\pi x\left[(x^2+1) - 5x^2\right]dx}$$

$$= 2\pi\int_0^{1/2} x\,\left[1-4x^2\right]dx$$

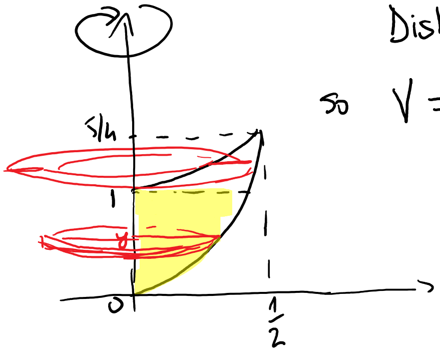
$$= 2\pi\int_0^{1/2} x - 4x^3\,dx$$

$$= 2\pi\left[\frac{1}{2}x^2 - x^4\right]\Big|_0^{1/2}$$

$$= 2\pi\left[\frac{1}{8} - \frac{1}{16}\right] = 2\pi\left[\frac{1}{16}\right]$$

$$= \boxed{\frac{\pi}{8}}$$

   (b) Using the washers/disks method, write the volume of this solid in terms of definite integrals. DO NOT EVALUATE the integrals.



Disks for $0 \le y \le 1$ & Washers on $1 \le y \le 5/4$

$$\text{so } V = \int_0^1 \pi\left(\sqrt{\tfrac{y}{5}}\right)^2 dy + \int_1^{5/4} \pi\left(\sqrt{\tfrac{y}{5}}\right)^2 - \pi\left(\sqrt{y-1}\right)^2 dy$$

$$= \boxed{\pi\int_0^1 \frac{y}{5}\,dy + \pi\int_1^{5/4}\left(\frac{y}{5}\right) - (y-1)\,dy}$$

OR

$$= \boxed{\pi\int_0^{5/4}\left(\frac{y}{5}\right)dy - \pi\int_1^{5/4}(y-1)\,dy}$$

$y = 5x^2 \Rightarrow x^2 = y/5$

$y = 1+x^2 \Rightarrow x^2 = y-1$

7. (a) (4 points) Set up a definite integral for the arclength of the curve

$$y = \sin(2x) \text{ for } 0 \leq x \leq \pi/4.$$

DO NOT EVALUATE THIS INTEGRAL.

$$\frac{dy}{dx} = \cos(2x) \cdot 2$$

$$L = \int_0^{\pi/4} \sqrt{1 + (2\cos(2x))^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + 4\cos^2(2x)}\, dx$$

(b) (6 points) Approximate the integral in part (a) using the Trapezoid Rule with $n = 3$ subintervals. Simplify your answer, but leave it in exact form, as an expression in terms of π and square roots, evaluating all trig functions.

$$\Delta x = \frac{\pi/4 - 0}{3} = \frac{\pi}{12} \Rightarrow x_0 = 0,\ x_1 = \frac{\pi}{12},\ x_2 = \frac{\pi}{6},\ x_3 = \frac{\pi}{4}$$

$$\int_0^{\pi/4} \sqrt{1 + 4\cos^2(2x)}\, dx \cong T_3$$

$$= \frac{\pi/12}{2}\left[\sqrt{1+4\cos^2(0)} + 2\sqrt{1+4\cos^2\left(2\tfrac{\pi}{12}\right)} + 2\sqrt{1+4\cos^2\left(2\tfrac{\pi}{6}\right)} + \sqrt{1+4\cos^2\left(2\tfrac{\pi}{4}\right)}\right]$$

$$= \frac{\pi}{24}\left[\sqrt{1+4(1)^2} + 2\sqrt{1+4\left(\tfrac{\sqrt{3}}{2}\right)^2} + 2\sqrt{1+4\left(\tfrac{1}{2}\right)^2} + \sqrt{1+4(0)^2}\right]$$

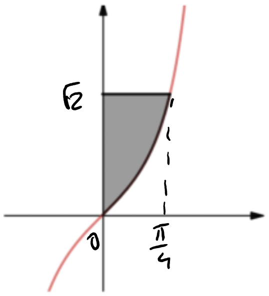
$$= \frac{\pi}{24}\left[\sqrt{5} + 2(2) + 2\sqrt{2} + 1\right]$$

$$= \boxed{\frac{\pi}{24}\left[\sqrt{5} + 2\sqrt{2} + 5\right]}$$

8. (10 points) The region shown below is bounded by the curves

$$y = \sec(x)\tan(x) \text{ and } y = \sqrt{2}, \text{ for } 0 \le x \le \pi/4.$$

Compute the **y-coordinate**, $\bar{y}$, for the center of mass of a thin lamina of uniform density occupying this region. *Show all work and box your answer. Give your answer in decimal form, rounded to 2 decimal digits.*



$$\text{Area } A = \int_0^{\pi/4} \left(\sqrt{2} - \sec x \tan x\right) dx$$

$$= \sqrt{2}\, x \Big|_0^{\pi/4} - \sec x \Big|_0^{\pi/4}$$

$$= \frac{\sqrt{2}\pi}{4} - \left(\sqrt{2} - 1\right) = \frac{\pi\sqrt{2}}{4} - \sqrt{2} + 1 \cong 0.6965$$

$$\bar{y} = \frac{1}{A}\int_0^{\pi/4} \frac{1}{2}(\sqrt{2})^2 - \frac{1}{2}\left(\sec x \tan x\right)^2 dx$$

$$= \frac{1}{A}\int_0^{\pi/4} 1 - \frac{1}{2}\sec^2 x \tan^2 x \, dx$$

$$= \frac{1}{A}\left[\; x \Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \tan^2 x \sec^2 x \, dx \right. \longrightarrow \boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array}}$$

$$= \frac{1}{A}\left[\frac{\pi}{4} - \frac{1}{2}\int_0^1 u^2 \, du\right] \longleftarrow$$

$$= \frac{1}{A}\left[\frac{\pi}{4} - \frac{u^3}{6}\Big|_0^1\right]$$

$$= \frac{1}{A}\left[\frac{\pi}{4} - \frac{1}{6}\right]$$

$$= \frac{\pi/4 - 1/6}{\left(\frac{\pi\sqrt{2}}{4} - \sqrt{2} + 1\right)} \cong \frac{0.61873}{0.6965} \cong \boxed{0.89}$$

9. (10 points) Solve the following differential equation, subject to the stated initial value. Show all steps, and give your answer in explicit form, $y = f(x)$.

$$(x^2-2x)y' = (x-4)y, \quad y(1) = -3$$

$$\frac{dy}{dx} = \left(\frac{x-4}{x^2-2x}\right)(y)$$

$$\int \frac{1}{y}\,dy = \int \frac{x-4}{x(x-2)}\,dx$$

$$\ln|y| = \int \frac{2}{x} + \frac{-1}{x-2}\,dx$$

Partial Fractions Decomposition

$$\frac{x-4}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$x-4 = A(x-2) + Bx$$

$$x=0: \quad -4 = A(-2) \Rightarrow A = 2$$

$$x=2: \quad -2 = B(2) \Rightarrow B = -1$$

$$\ln|y| = 2\ln|x| - \ln|x-2| + C$$

$$\ln|y| = \ln\left(\frac{x^2}{|x-2|}\right) + C$$

$$|y| = \cancel{e}^{\cancel{\ln}\left(\frac{x^2}{|x-2|}\right)} \cdot e^{c}$$

$$y = A\,\frac{x^2}{|x-2|} \qquad (A = \pm e^{c})$$

Initial condition: $y(1) = -3 \Rightarrow A = -3 \Rightarrow \boxed{y = \frac{-3x^2}{|x-2|}}$ & $\cdot\ y(1) = -3$

Note: Since y has a vertical asymptote at $x=2$ and the given condition is at $x = 1 < 2$, the solution need only be valid for $x < 2$. On this domain, $|x-2| = 2-x$ so:

Solution: $\boxed{y = \frac{-3x^2}{2-x} = \frac{3x^2}{x-2}, \text{ for } x<2}$

10. A large vat contains 10 liters of brine (salt dissolved in water). More brine is pumped into the vat at a rate of 2 liters per hour. The incoming brine solution contains 3 grams of salt per liter. The solution in the vat is kept thoroughly mixed and is drained from the vat at a rate of 2 liters per hour.

(a) (3 points) Set up a **differential equation** for the amount $y = y(t)$ of grams of salt in the vat at t hours. Do not solve yet.

$$\frac{dy}{dt} = \left(\frac{dy}{dt}\right)_{in} - \left(\frac{dy}{dt}\right)_{out} = (2\,L/hr)(3\,gr/L) - (2\,L/hr)\left(\frac{y}{10}\,gr/L\right)$$

$$\boxed{\frac{dy}{dt} = 6 - \frac{y}{5}}\ (gr/hr)$$

(b) (5 points) Denote by y_0 the intial amount of salt in the vat, in grams, at time $t = 0$ hours. Solve the differential equation to find $y(t)$. Your answer will include the unknown constant y_0.

$$\frac{dy}{dt} = \frac{30-y}{5} \Rightarrow \int \frac{1}{30-y}\,dy = \int \frac{1}{5}\,dt$$

$$-\ln|30-y| = \frac{1}{5}t + C$$

$$\ln|30-y| = -1/5\,t + C_1$$

$$|30-y| = e^{-1/5t} \cdot e^{C_1}$$

$$30-y = A e^{-1/5t} \qquad (A = \pm e^{C_1})$$

$$y = 30 - Ae^{-1/5t}$$

at $t=0$, $y=y_0$: $y_0 = 30 - A \Rightarrow A = 30 - y_0$

$$\boxed{y = 30 - (30-y_0)e^{-1/5t}}$$

or: $$\boxed{y = 30 + (y_0-30)e^{-1/5t}}$$

(c) (2 points) Suppose that after 4 hours, the concentration of the salt in the vat is 4 grams per liter. What was the initial concentration of the salt in the vat (in grams per liter)?

After $t=4$ hrs, there are $y(4) = 30 + (y_0-30)e^{-4/5}$ grams of salt in the tank

$$\text{concentration} = \frac{\text{amt of salt}}{\text{volume in tank}} = \frac{30+(y_0-30)e^{-4/5}}{10} = 4\ \text{grams}/L$$

$$\Rightarrow 30 + (y_0-30)e^{-4/5} = 40 \Rightarrow (y_0-30)e^{-4/5} = 10$$

$$\Rightarrow y_0 - 30 = 10e^{4/5} \Rightarrow y_0 = 30 + 10e^{4/5}$$

$$\text{Initial concentration} = \frac{y_0}{10} = \frac{30+10e^{4/5}}{10} = \boxed{3 + e^{4/5} \cong 5.23\ gr/L}$$