1. (14 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int \sqrt{x}\ln(\sqrt{x})dx$

$\boxed{u=\sqrt{x}} \Rightarrow x=u^2$, $\boxed{dx = 2u\,du}$

$= \int u \ln(u) \cdot 2u\,du$

$= 2\int u^2 \ln(u)\,du$ IBP: $w = \ln u$, $dv = u^2\,du$; $dw = \frac{1}{u}du$, $v = \frac{1}{3}u^3$

$= 2\left[\frac{1}{3}u^3 \ln u - \int \frac{1}{3}u^{\cancel{3}\,2} \frac{1}{\cancel{u}}\,du\right]$

$= 2\left[\frac{1}{3}u^3 \ln u - \frac{1}{9}u^3\right] + C = \boxed{\frac{2}{3}x^{3/2}\ln\sqrt{x} - \frac{2}{9}x^{3/2} + C}$

$= \boxed{\frac{1}{3}x^{3/2}\ln x - \frac{2}{9}x^{3/2} + C}$

Remark:

There are alternative ways to proceed.
For instance, one could use the fact that $\ln\sqrt{x} = \frac{1}{2}\ln x$ at the start, skip the u-sub. and use IBP w/ $u = \ln x$, $dv = \sqrt{x}\,dx$

(b) $\int \frac{1}{x^4\sqrt{x^2-4}}dx$

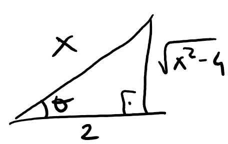
$\left[\begin{array}{l} x = 2\sec\theta \\ dx = 2\sec\theta\tan\theta\,d\theta \end{array}\right]$

$= \int \frac{1}{(2^4\sec^4\theta)(\cancel{2}\cancel{\tan\theta})}\cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta = \frac{1}{16}\int \frac{1}{\sec^3\theta}\,d\theta$

$= \frac{1}{16}\int \cos^3\theta\,d\theta = \frac{1}{16}\int (1-\sin^2\theta)\cos\theta\,d\theta$ $\left[\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}\right]$

$= \frac{1}{16}\int (1-u^2)\,du = \frac{1}{16}\left(u - \frac{1}{3}u^3\right) + C$

$= \frac{1}{16}\sin\theta - \frac{1}{48}\sin^3\theta + C$



$= \boxed{\frac{1}{16}\frac{\sqrt{x^2-4}}{x} - \frac{1}{48}\left(\frac{\sqrt{x^2-4}}{x}\right)^3 + C}$

2. (14 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\displaystyle\int_0^{\pi/4} \frac{\sin^4\theta}{\cos^6\theta}\,d\theta$ $\displaystyle = \int_0^{\pi/4} \left(\frac{\sin\theta}{\cos\theta}\right)^4 \frac{1}{\cos^2\theta}\,d\theta$

$$= \int_0^{\pi/4} \tan^4\theta \underbrace{\sec^2\theta\, d\theta} \qquad \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix}$$

$$= \int_0^1 u^4 \, du$$

$$= \frac{1}{5}u^5\Big|_0^1 = \boxed{\frac{1}{5}}$$

(b) $\displaystyle\int_0^{1/2} \frac{\sqrt{2x}}{2x-4}\,dx$ $\qquad [u = \sqrt{2x} \Rightarrow u^2 = 2x \Rightarrow \cancel{2}u\,du = \cancel{2}dx]$

$$= \int_0^1 \frac{u}{u^2-4}\, u\,du = \int_0^1 \frac{u^2}{u^2-4}\,du = \int_0^1 \frac{(u^2-4)+4}{u^2-4}\,du$$

$$= \int_0^1 1 + \frac{4}{u^2-4}\,du$$

Partial Fractions:

$$\frac{4}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$4 = A(u+2) + B(u-2)$$

$$4 = (A+B)u + 2A - 2B$$

$$\begin{cases} A+B=0 \\ 2A-2B=4 \end{cases} \Rightarrow \begin{cases} B=-A \\ 4A=4 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$= \int_0^1 1 + \frac{1}{u-2} + \frac{-1}{u+2}\,du$$

$$= \Big[u + \ln|u-2| - \ln|u+2|\Big]\Big|_0^1$$

$$= 1 + \underbrace{\cancel{\ln}1}_{0} - \ln 3 - 0 - \cancel{\ln}2 + \cancel{\ln}2$$

$$= \boxed{1 - \ln 3}$$

3. (10 points) Compute the area of the region that is bounded above by $y = \sqrt{8x}$ and below by the line $y = 1$ and the curve $y = x^2$. Simplify and box your answer.

INTERSECTIONS:

$y=1$ w/ $y=x^2$ at $(1,1)$

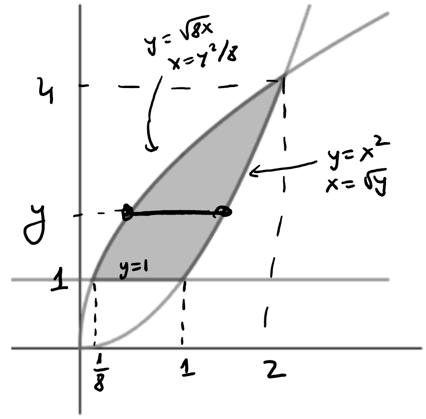
$y=1$ w/ $y=\sqrt{8x}$ at $(\frac{1}{8},1)$

$y=x^2$ w/ $y=\sqrt{8x}$: $x^2=\sqrt{8x}$

$x^4 = 8x$

$x(x^3-8)=0$

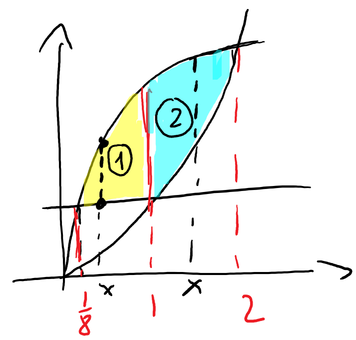
$x=0, x=2 \Rightarrow (2,4)$



EASIER SETUP (in y)

$$A = \int_1^4 \Big(\underset{\text{Right}}{\sqrt{y}} - \underset{\text{Left}}{\frac{y^2}{8}}\Big)\,dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{24}y^3\right)\Big|_1^4 = \left(\frac{2}{3}\cdot 8 - \frac{64}{24}\right) - \left(\frac{2}{3} - \frac{1}{24}\right) = \boxed{\frac{49}{24}}$$

(square units)



SET UP in x (requires 2 integrals)

$$A = \underbrace{\int_{1/8}^{1} (\overset{\text{Top}}{\sqrt{8x}} - \overset{\text{Bottom}}{1})\,dx}_{1} + \underbrace{\int_1^2 (\overset{\text{Top}}{\sqrt{8x}} - \overset{\text{Bottom}}{x^2})\,dx}_{2}$$

$$= \left(\frac{2}{3}(8x)^{3/2}\cdot\frac{1}{8} - x\right)\Big|_{1/8}^{1} + \left(\frac{2}{3}(8x)^{3/2}\frac{1}{8} - \frac{1}{3}x^3\right)\Big|_1^2 = \boxed{\frac{49}{24}}$$

4. (14 points) One model for air resistance predicts that a particular ball thrown straight up in the air will have velocity at t seconds given by:

$$v(t) = ce^{-t} - 10 \text{ meters/sec, for some constant } c,$$

where upward is considered positive velocity. Assume the ball is thrown straight upward starting from the ground with an initial velocity of 20 m/s.

(a) Find the formula for the height $h(t)$ of the ball after t seconds.

To compute c: $v(0) = 20 \Rightarrow ce^0 - 10 = 20 \Rightarrow c = 30$.

So: $v(t) = 30e^{-t} - 10$

Integrating: $h(t) = 30(-e^{-t}) - 10t + D$, $h(0) = 0 \Rightarrow D = 30$

$\therefore \boxed{h(t) = -30e^{-t} - 10t + 30}$ meters.

(b) Find the **total distance** traveled by the ball from $t = 0$ to $t = 2$ seconds. You may give your final answer as a decimal accurate to 3 digits after the decimal point (or leave in exact form).

We want: $\int_0^2 |v(t)|\,dt = \int_0^2 |30e^{-t} - 10|\,dt$

The velocity changes sign when: $v(t) = 30e^{-t} - 10 = 0 \Rightarrow e^{-t} = \frac{1}{3} \Rightarrow -t = \ln(\frac{1}{3})$

so at $t = \ln 3$, which is in between the $[0,2]$ bounds.

For $0 \le t < \ln 3$: $v(t) = 30e^{-t} - 10 > 0$, then, for $t > \ln 3$, $v(t) < 0$.

$\therefore$ Total distance $= \int_0^{\ln 3} 30e^{-t} - 10\,dt + \int_{\ln 3}^2 10 - 30e^{-t}\,dt$

$= \left[-30e^{-t} - 10t\right]\Big|_0^{\ln 3} + \left[10t + 30e^{-t}\right]\Big|_{\ln 3}^2$

$= \left[-30e^{-\ln 3} - 10\ln 3 + 30e^0\right] + \left[20 + 30e^{-2} - 10\ln 3 - 30e^{-\ln 3}\right]$

$= -\cancel{30\frac{1}{3}} - 10\ln 3 + 30 + \cancel{20} + \frac{30}{e^2} - 10\ln 3 - \cancel{30\frac{1}{3}}$

$= \boxed{30 - 20\ln 3 + \frac{30}{e^2}} \approx 12.088$ meters

5. (14 points) A particle is sliding down the curve $y = 10 - x^3$. At time $t = 0$ the particle starts at $(0, 10)$. The x-coordinate of the particle at time t is $x(t) = \frac{t}{3}$. Time is measured in seconds, distance in meters. Let $a(t)$ denote the arclength distance traveled by the particle along the curve in the first t seconds.

(a) Set up an integral expression equal to $a(t)$. Do NOT attempt to evaluate it.

$y = 10 - x^3$
$dy/dx = -3x^2$

$$a(t) = \int_{x(0)}^{x(t)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{t/3} \sqrt{1 + (-3x^2)^2}\, dx = \int_0^{t/3} \sqrt{1 + 9x^4}\, dx \quad \underline{\text{meters}}$$

(b) Calculate $a'(1)$. Include units.

Using FTC I (& Chain Rule): $a'(t) = \frac{d}{dt}\int_0^{t/3} \sqrt{1+9x^4}\,dx = \sqrt{1 + 9\left(\frac{t}{3}\right)^4} \cdot \frac{1}{3}$

$= \frac{1}{3}\sqrt{1 + \cancel{9}\frac{t^4}{\cancel{81}9}} = \frac{1}{9}\sqrt{9 + t^4}$

Evaluating at 1: $\boxed{a'(1) = \frac{1}{9}\sqrt{10}}$, in $\boxed{\text{meters/sec.}}$

(c) Use Simpson's Rule with $n = 4$ subdivisions to approximate the value of $a(1)$. Show work, and give your answer correct to 3 decimal places.

First: set up the integral to approximate: $a(1) = \int_0^{1/3} \sqrt{1 + 9x^4}\, dx$

We set: $\Delta x = \frac{b-a}{n} = \frac{1/3}{4} = \frac{1}{12}$ & the endpoints are: $0, \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{1}{3}$

$$\therefore a(1) \cong S_4 = \frac{1/12}{3}\left[\sqrt{1 + 9(0)^4} + 4\sqrt{1 + 9\left(\frac{1}{12}\right)^4} + 2\sqrt{1 + 9\left(\frac{1}{6}\right)^4} + 4\sqrt{1 + 9\left(\frac{1}{4}\right)^4} + \sqrt{1 + 9\left(\frac{1}{3}\right)^4}\right]$$

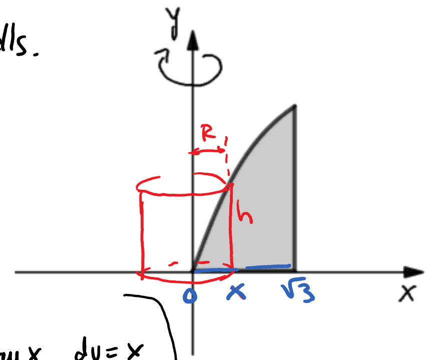
$\cong \boxed{0.337}$

6. (10 points) Let $\mathcal{R}$ be the region in the first quadrant bounded by:

the x-axis, the curve $y = 3\arctan(x)$ and the line $x = \sqrt{3}$.

Calculate the volume of the solid of revolution generated by rotating $\mathcal{R}$ about the y-axis.

This can be done via either washers or shells.



SHELLS: $\int_0^{\sqrt{3}} 2\pi R h \, dx$

$= \int_0^{\sqrt{3}} 2\pi \, x \, (3\arctan x) \, dx$

$= 6\pi \int_0^{\sqrt{3}} x \arctan x \, dx$

IBP: $u = \arctan x$, $dv = x$; $du = \frac{1}{1+x^2}$, $v = \frac{1}{2}x^2$

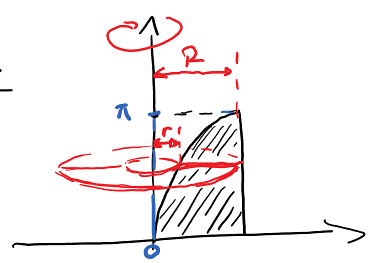
$= 6\pi \left[\frac{1}{2} x^2 \arctan x \Big|_0^{\sqrt{3}} - \frac{1}{2} \int_0^{\sqrt{3}} \frac{x^2}{1+x^2} \, dx \right]$

$= 3\pi \left[x^2 \arctan x \Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} 1 - \frac{1}{1+x^2} \, dx \right]$

$= 3\pi \left[x^2 \arctan x - x + \arctan x \right] \Big|_0^{\sqrt{3}}$

$= 3\pi \left[\cancel{3} \frac{\pi}{\cancel{3}} - \sqrt{3} + \frac{\pi}{3} \right] = 3\pi \left[\frac{4\pi}{3} - \sqrt{3} \right] = \boxed{4\pi^2 - 3\sqrt{3}\pi} \cong 23.15$

WASHERS:



upper bound for y is $y = 3\arctan\sqrt{3} = 3 \frac{\pi}{3} = \pi$

$V = \int_0^{\pi} \pi R^2 - \pi r^2 \, dy$

$= \int_0^{\pi} \pi (\sqrt{3})^2 - \pi \left(\tan^2(y/3) \right) dy$

$= \pi \int_0^{\pi} 3 - \left(\sec^2(y/3) - 1 \right) dy$

$= \pi \int_0^{\pi} 4 - \sec^2(y/3) \, dy.$

$= \pi \left[4y - 3\tan(y/3) \right] \Big|_0^{\pi}$

$= \boxed{4\pi^2 - 3\pi\sqrt{3}}$ (cubic units)

7. (10 points) Find the solution to the differential equation

$$\pi \frac{dy}{dx} = \frac{e^{x-y}}{\sqrt{4-e^{2x}}}$$

that satisfies the initial condition $y(0) = 0$. Give your solution in explicit form, $y = f(x)$.

Separating the variables: $\pi \frac{dy}{dx} = \left(\frac{e^x}{\sqrt{4-e^{2x}}}\right)\cdot\left(e^{-y}\right)$

and integrating: $\int \pi e^y \, dy = \int \frac{e^x}{\sqrt{4-e^{2x}}} \, dx$ $\begin{pmatrix} u = e^x \\ du = e^x dx \end{pmatrix}$

$$= \int \frac{1}{\sqrt{4-u^2}} \, du$$

$$= \arcsin\left(\frac{u}{2}\right) + C$$

$$\therefore \pi e^y = \arcsin\left(\frac{e^x}{2}\right) + C$$

$$e^y = \frac{1}{\pi} \arcsin\left(\frac{e^x}{2}\right) + C_1$$

Using $y(0)=0$ to compute the constant of integration: $e^0 = \frac{1}{\pi} \arcsin\left(\frac{1}{2}\right) + C_1$

$$1 = \frac{1}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{6} + C_1$$

$$C_1 = 5/6.$$

$$\therefore e^y = \frac{1}{\pi} \arcsin\left(\frac{e^x}{2}\right) + \frac{5}{6}$$

Solving for y: $\boxed{y = \ln\left(\frac{1}{\pi} \arcsin\left(\frac{e^x}{2}\right) + \frac{5}{6}\right)}$

8. (14 points) At time $t = 0$ minutes a tank holds an initial volume $V_0 = 100$ m^3 of salty water, with an initial amount $S_0 = 3$ kg of salt dissolved in it.

Fresh water enters the tank at a rate of $10 + 2t$ m^3 per minute, where t is the time in minutes.

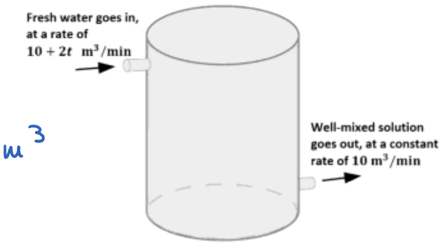
The salt always remains uniformly mixed throughout the water solution in the tank, and the solution exits the tank at a constant rate of 10 m^3/min.

(a) Find a formula for the volume $V(t)$ of salty water solution in the tank at time t minutes.

Given: $\frac{dV}{dt} = (10+2t) - (10) = 2t$ m^3/min

we integrate to get $V(t) = t^2 + C$ m^3

Since $V(0) = 100$ m^3, $\boxed{V(t) = t^2 + 100}$ m^3



(b) Set up a differential equation for $S(t)$, which is the amount (in kg) of salt in the tank at t min. Do not solve it yet.

$$\frac{dS}{dt} = (\text{rate in}) - (\text{rate out}) = 0 - \frac{S(t)\text{ kg}}{V(t)\text{ m}^3} \cdot 10\text{ m}^3/\text{min}$$

"fresh water" => no salt comes in.

$$\therefore \boxed{\frac{dS}{dt} = \frac{-10S}{t^2+100}} \quad (\text{kg}/\text{min})$$

(c) Solve the differential equation in part (b) to find a formula for $S(t)$.

$$\int \frac{1}{S}\,dS = \int \frac{-10}{t^2+100}\,dt$$

$$\ln S = -\cancel{10}\,\frac{1}{\cancel{10}}\arctan\left(\frac{t}{10}\right) + C$$

$$S = e^{-\arctan(t/10)+C} = C_1 e^{-\arctan(t/10)} \quad (\text{w}/\ C_1 = e^C)$$

$$S(0) = 3\text{kg} \Rightarrow 3 = C_1 e^{-\arctan(0)} \Rightarrow C_1 = 3$$

$$\boxed{S(t) = 3e^{-\arctan(t/10)}}\ \text{kg}$$

(d) How much salt is left in the tank after 10 min? Leave your answer in exact form.

$$S(10) = 3e^{-\arctan(10/10)}$$

$$= \boxed{3e^{-\pi/4}}\ \text{kg}$$