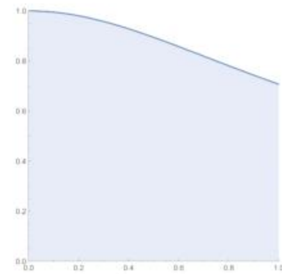
Solutions

1. (10 points) Find the center of mass $(\bar{x}, \bar{y})$ of a metal plate bounded by the graph of the function $f(x) = \dfrac{1}{\sqrt{1+x^2}}$ on the top, the x-axis on the bottom, the y-axis on the left and the line $x = 1$ on the right.
   Show all work and box your answer.



Solution: To calculate the mass, we use inverse trigonometric substitution $x = \tan\theta$, $dx = \sec^2\theta d\theta$, with the new limits 0 and $\pi/4$.

$$M = \int_0^1 \frac{1}{\sqrt{1+x^2}}dx = \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta d\theta = \int_0^{\pi/4} \sec\theta d\theta = \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4}$$
$$= \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \boxed{\ln\left(1+\sqrt{2}\right)}$$

$$M_x = \frac{1}{2}\int_0^1 \frac{1}{1+x^2}dx = \frac{1}{2}\arctan x\Big|_0^1 = \frac{1}{2}\cdot\frac{\pi}{4} = \boxed{\pi/8}$$

To calculate the moment M_y, we use the substitution $u = 1+x^2$, $du = 2xdx$, with the new limits $u = 1$ and $u = 2$.

$$M_y = \int_0^1 \frac{x}{\sqrt{1+x^2}}dx = \int_1^2 \frac{1}{\sqrt{u}}\cdot\frac{1}{2}du = \sqrt{u}\Big|_1^2 = \boxed{\sqrt{2}-1}$$

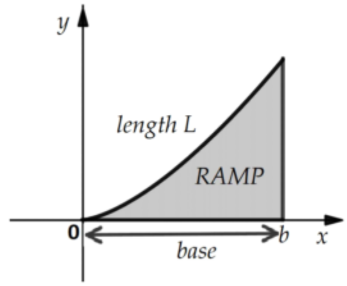
The center of mass is

$$(\bar{x}, \bar{y}) = (M_y/M, M_x/M) = \left(\frac{\sqrt{2}-1}{\ln\left(1+\sqrt{2}\right)}, \frac{\pi/8}{\ln\left(1+\sqrt{2}\right)}\right) = \boxed{\left(\frac{\sqrt{2}-1}{\ln\left(1+\sqrt{2}\right)}, \frac{\pi}{8\ln\left(1+\sqrt{2}\right)}\right)}$$

2. (10 points) While staying home in quarantine, you decide to build yourself a skateboard ramp.

   The top of the ramp is in the shape of the function $y = (4/3)x^{3/2}$, where x is the horizontal distance along the base of the ramp, in meters.

   If you want the top of the ramp to have an arclength $L = 2$ meters, how long will the base b have to be?



$$L = \int_0^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^b \sqrt{1+(2\sqrt{x})^2}\,dx$$

$$= \int_0^b \sqrt{1+4x}\,dx \qquad u = 1+4x, \quad du = 4dx$$

$$= \int_1^{1+4b} \sqrt{u}\ \frac{1}{4}\,du$$

$$= \frac{1}{\cancel{4}_2}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_1^{1+4b} = \frac{1}{6}\left(\left(\sqrt{1+4b}\right)^3 - 1\right)$$

$$L = 2 \implies \frac{1}{6}\left((1+4b)^{3/2}-1\right) = 2$$

$$(1+4b)^{3/2} - 1 = 12$$

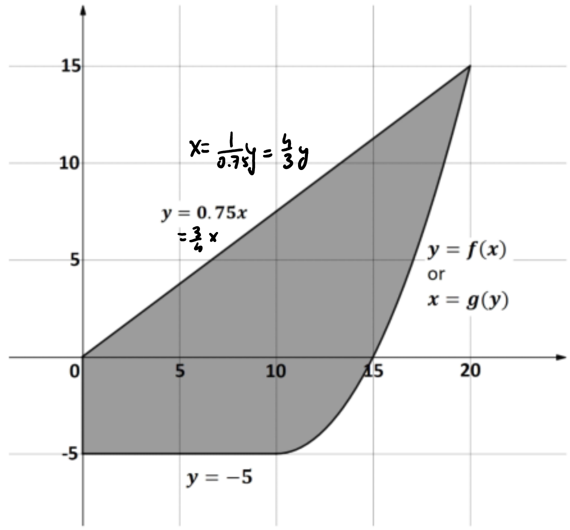
$$(1+4b)^{3/2} = 13$$

$$1+4b = 13^{2/3}$$

$$4b = 13^{2/3} - 1$$

$$\boxed{b = \frac{13^{2/3}-1}{4}} \cong 1.132 \text{ meters}$$

3. (10 points) The shaded region shown is bounded on its top by the line $y = 0.75x$, and on its bottom by $y = -5$ for $0 \le x \le 10$ and by some non-linear function $y = f(x)$ (with inverse $x = g(y)$) for $10 \le x \le 20$.



(a) Set up an integral expression in x equal to the area of this region.
Your answer will include $f(x)$. Do not simplify or compute, and no need to justify.

$$A = \int_0^{10} (0.75x - (-5))\,dx + \int_{10}^{20} (0.75x - f(x))\,dx$$

$$= 87.5 + \int_{10}^{20} (0.75x - f(x))\,dx$$

(b) Set up an integral expression in y equal to the area of this region.
Your answer will include $g(y)$. Do not simplify or compute, and no need to justify.

$$A = \int_{-5}^{0} g(y)\,dy + \int_0^{15} \left(g(y) - \overset{4/3}{\frac{1}{0.75}}y\right)dy$$

4. (10 points) Archeologists found an ancient circular mound created by soil dug up from the center of the structure. The mound is a solid of revolution obtained by rotating the shaded area shown below around the line at the center. Measurements show that the top of the shaded section of the mound is approximately given by the function:

$$f(x) = 2\sin^2\left(\frac{\pi x^2}{100}\right), \quad 10 \le x \le 10\sqrt{2}$$

where all dimensions are in meters, and x is the distance from the center.

Compute the volume of the mound. Show all work and leave your answer in exact simplified form.

Solution: We will use the shell method. The volume is given by

$$\int_{10}^{10\sqrt{2}} 2\pi x f(x) dx$$

$$= \int_{10}^{10\sqrt{2}} 2\pi x 2(\sin(\pi x^2/100))^2 dx$$

We use the substitution $u = \pi x^2/100$, $du = (2\pi x/100)dx$.

The new limits of integration are π and 2π.

$$\int_{10}^{10\sqrt{2}} 2\pi x 2(\sin(\pi x^2/100))^2 dx = \int_{\pi}^{2\pi} 200(\sin u)^2 du$$

$$= \int_{\pi}^{2\pi} 200(1/2)(1-\cos(2u))du$$

$$= 100(u-(1/2)\sin(2u))\Big|_{u=\pi}^{u=2\pi} = \boxed{100\pi\ m^3}$$

5. (10 points) A leaky bucket weighs 2 kg and initially contains 20 kg of water.

   It is being pulled up on a rope at the speed of 2 m/s while losing 1 kg of water per second. Ignore the weight of the rope and assume that the gravitational acceleration is 9.8 m/s^2.

   The work done to lift the bucket to the height h above the initial level was 823.2 Joules. Find the height h.

   Show all your work and box the final answer.

Solution:

The initial mass of the bucket with water was 22 kg. For every meter in elevation gain, 0.5 kg of water were lost so the mass of the bucket with water was $22 - x/2$ kilograms at the level x.

The weight at this level was $9.8(22 - x/2)$ N.

The work done to lift the bucket to level h was

$$\int_0^h 9.8(22 - x/2)dx$$

$$= 9.8(22x - x^2/4)\Big|_0^h = 9.8(22h - h^2/4).$$

According to the problem, the work done was $823.2 = 9.8(22h - h^2/4)$.

Solving for h, we obtain $h = 4$ or $h = 84$.

The solution $h = 84$ has to be rejected because the bucket would have lost all water at the height of 40 meters and the formula for the mass would no longer apply. Hence, $\boxed{h = 4}$ meters.

6. (10 points) In the year 2020 (which we take to be $t = 0$) there are 5000 wolves in a certain forest.

In the absence of hunting, the wolf population would increase at the rate of 1% per year. However, hunters are killing wolves at the steady rate of 100 wolves per year.

Let $W(t)$ represent the wolf population in this forest, t years after 2020.

(a) Write a differential equation that $W(t)$ satisfies.

2pts

$$\frac{dW}{dt} = 0.01W - 100 = 0.01(W - 10{,}000)$$

(b) Solve this differential equation to find a formula for $W(t)$ in terms of t. Show all steps.

6pts

$$\int \frac{1}{W - 10{,}000}\, dW = \int 0.01\, dt$$

$$\ln|W - 10{,}000| = 0.01t + C$$

$$|W - 10{,}000| = e^{0.01t} \cdot e^{C} \leftarrow c_1$$

$$W = 10{,}000 \pm c_1 e^{0.01t} \qquad (c_2 = \pm c_1)$$

$$W = 10{,}000 + c_2 e^{0.01t}$$

$$W(0) = 5000 \Rightarrow \quad 5000 = 10{,}000 + c_2 \Rightarrow c_2 = -5000$$

$$\therefore \boxed{W = 10{,}000 - 5000e^{0.01t}}$$

(c) In what year will the entire wolf population be exterminated from this forest area? Round your answer to the nearest year. Your answer should be some year in this century.

2pts

$$W = 0 = 10{,}000 - 5000e^{0.01t}$$

$$e^{0.01t} = \frac{10{,}000}{5{,}000} = 2$$

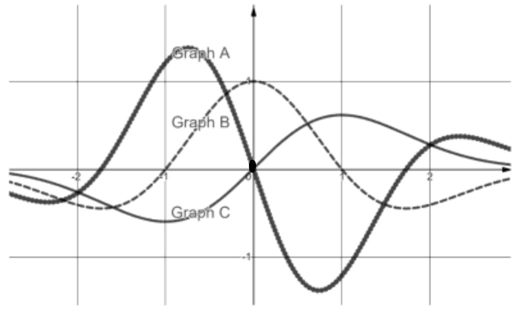
$$0.01t = \ln 2$$

$$t = 100 \ln 2 \cong 69.3147\ldots \text{ years after 2020}$$

$$\therefore \text{in the year } \boxed{2089}$$

7. (10 points) (a) The following picture shows the graphs of three functions, labeled A, B, and C.

3pts



For each of (i)-(iii) below, list all correct answers among the graphs A, B, C, or state "none". No need to justify.

$A = f'$

i. Which of A-C could be the graph of an antiderivative of Graph A? $A = B' \Rightarrow$ $\boxed{\text{Graph B}}$.

ii. Which of A-C could be the graph of an antiderivative of Graph B? $B = C' \Rightarrow$ $\boxed{\text{Graph C}}$

iii. Which of A-C could be graph of an antiderivative of Graph C? $C = (?)'$ $\boxed{\text{none}}$.

(b) Which of the following expressions $y = f(x)$ are solutions of the differential equation:

$$\frac{dy}{dx} = e^{x^2} \text{ satisfying } f(1) = 2?$$

7pts

Circle the correct ones, and cross out the incorrect ones. For the ones that you crossed out, show which of the conditions it fails to satisfy: $\frac{dy}{dx} = e^{x^2}$, $f(1) = 2$, or both.

~~(A)~~ $y = \int_1^x e^{t^2}\,dt$ By FTC I: $\frac{dy}{dx} = e^{x^2}$, but $f(1) = 0 \neq 2$

(B) $y = \int_1^x e^{t^2}\,dt + 2$ $\frac{dy}{dx} = e^{x^2}$ ✓ $f(1) = \int_1^1 e^{t^2}dt + 2 = 0 + 2 = 2$ ✓

~~(C)~~ $y = \int_1^{x^2} e^{t}\,dt$ $\frac{dy}{dx} = e^{x^2} \cdot 2x \neq e^{x^2}$ $f(1) = 0$ BOTH FAIL

~~(D)~~ $y = \int_1^x (e^{t^2} + 2)\,dt$ $\frac{dy}{dx} = e^{x^2} + 2 \neq e^{x^2}$ $f(1) = 0$ BOTH FAIL

~~(E)~~ $y = \underbrace{\frac{1}{2x}e^{x^2}}_{\text{Prod. Rule (or Quotient Rule)}} - \frac{1}{2}e + 2$ $\frac{dy}{dx} \neq e^{x^2}$ $f(1) = \frac{1}{2}e - \frac{1}{2}e + 2 = 2$

8. (10 points) Evaluate the following integral. Show all steps and use the methods of this class. Credit will depend on clarity and completeness of the solution steps shown.

$$\int \operatorname{arcsec}\left(\sqrt{x^2+4x+5}\right)\, dx$$

x^2+4x+5
$= (x+2)^2+1$

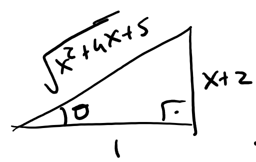
$$\int \operatorname{arcsec} \sqrt{x^2+4x+5}\, dx$$
$$= \int \operatorname{arcsec} \sqrt{(x+2)^2+1}\, dx$$

$x+2 = \tan\theta$
$dx = \sec^2\theta\, d\theta$

$$= \int \operatorname{arcsec}\left(\sqrt{\tan^2\theta+1}\right) \sec^2\theta\, d\theta$$
$$= \int \operatorname{arcsec}(\sec\theta)\cdot \sec^2\theta\, d\theta$$
$$= \int \theta \sec^2\theta\, d\theta$$

IBP $u = \theta$ $\quad dv = \sec^2\theta\, d\theta$
$du = d\theta$ $\quad v = \tan\theta$

$$= \theta\tan\theta - \int \tan\theta\, d\theta$$
$$= \theta\tan\theta - \ln|\sec\theta| + C$$

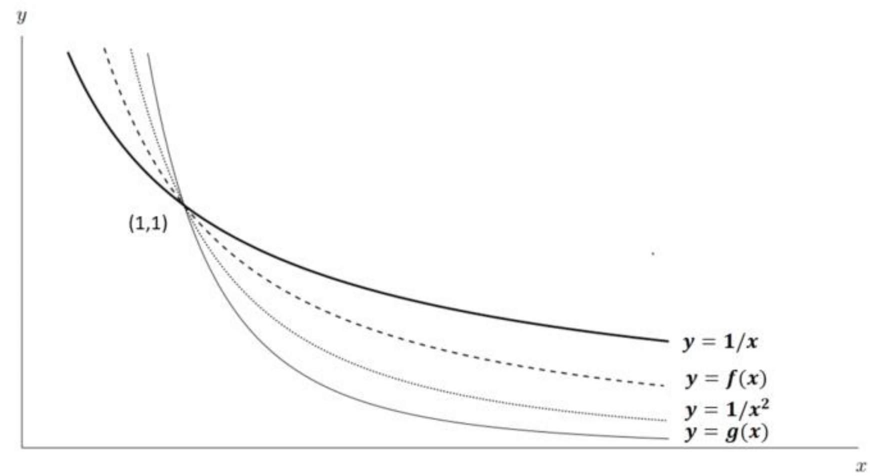


$\tan\theta = \frac{x+2}{1}$

$\therefore \sec\theta = \frac{\sqrt{x^2+4x+5}}{1}.$

$$= \arctan(x+2)\cdot(x+2) - \ln\sqrt{x^2+4x+5} + C$$
$$= \boxed{(x+2)\arctan(x+2) - \ln\sqrt{x^2+4x+5} + C}$$

9. (10 points) Consider the graphs below, depicting four positive, continuous, and decreasing functions. The functions cross only at at the point $(1, 1)$.



Use the graph and the Comparison Test to determine whether the following improper integrals converge or diverge. Circle the appropriate answer. If there is not enough information to determine convergence or divergence, circle "not enough information". **Justify your answers.**

(a) $\int_1^\infty g(x)dx$

CONVERGES DIVERGES NOT ENOUGH INFORMATION

$g(x) < \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}dx$ converges

(b) $\int_1^\infty \frac{1}{g(x)}dx$

CONVERGES DIVERGES NOT ENOUGH INFORMATION

$g(x) < \frac{1}{x^2} \Rightarrow \frac{1}{g(x)} > x^2 \Rightarrow \int_1^\infty \frac{1}{g(x)}dx > \int_1^\infty x^2 dx = \infty$

(c) $\int_1^\infty \sqrt{f(x)}dx$

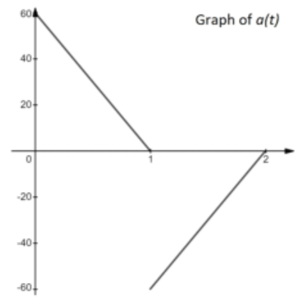
CONVERGES DIVERGES NOT ENOUGH INFORMATION

$\frac{1}{x^2} < f(x) < \frac{1}{x} \overset{(x>0)}{\Rightarrow} \frac{1}{x} < \sqrt{f(x)} < \frac{1}{\sqrt{x}} \Rightarrow \underbrace{\int_1^\infty \frac{1}{x}dx}_{\infty} < \int_1^\infty \sqrt{f(x)}dx < \int_1^\infty \frac{1}{\sqrt{x}}dx$

10. (10 points) In a 2-second laboratory experiment, a particle moves in a straight line, while its acceleration is manipulated by a force field. The resulting acceleration function (in m/s^2) is a multi-part function, whose expression and graph are shown below.

$$a(t) = \begin{cases} 60(1-t) & 0 \le t < 1 \\ 60(t-2) & 1 \le t < 2 \end{cases}$$

$$= \begin{cases} 60-60t & 0 \le t < 1 \\ 60t-120 & 1 \le t < 2 \end{cases}$$



Assume that the particle begins at rest at $t = 0$ seconds.

(a) Compute the expression in terms of t for the velocity of the particle from $t = 0$ to $t = 1$ seconds.

$$V_1(t) = \int 60-60t\,dt = 60t - 30t^2 + C$$
$$V_1(0) = 0 \Rightarrow C = 0$$
$$\Rightarrow \boxed{V_1(t) = 60t - 30t^2} \text{ on } [0,1]$$

(b) Compute the expression in terms of t for the velocity of the particle from $t = 1$ to $t = 2$ seconds.

$$V_2(t) = \int (60t - 120)\,dt = 30t^2 - 120t + D$$
$$V_2(1) = V_1(1) = 60 - 30 = 30$$
$$\Rightarrow 30 - 120 + D = 30 \quad \therefore D = 120$$

$$\boxed{V_2(t) = 30t^2 - 120t + 120} \text{ on } [1,2]$$

(c) Compute the average velocity at which the particle was moving from $t = 0$ to $t = 2$ seconds. Show work.

$$V_{ave} = \frac{1}{2}\int_0^2 v(t)\,dt = \frac{1}{2}\left[\int_0^1 (60t - 30t^2)\,dt + \int_1^2 (30t^2 - 120t + 120)\,dt\right]$$
$$\frac{1}{2} = \left[(30t^2 - 10t^3)\Big|_0^1 + (10t^3 - 60t^2 + 120t)\Big|_1^2\right]$$
$$= \frac{1}{2}\left[(30-10) + (80 - \cancel{240} + \cancel{240}) - (10 - 60 + 120)\right]$$
$$= \frac{1}{2}\left[20 + 10\right] = \boxed{15 \text{ m/s}}$$