1. (14 points) Evaluate the following indefinite integrals. Show work and box your answers.

(a) $\int \frac{x^2+1}{x^2-2x-3}\,dx = \int 1 + \frac{2x+4}{x^2-2x-3}\,dx$ (polynomial long division or algebra)

$= \int 1 + \frac{2x+4}{(x-3)(x+1)}\,dx$

$= \int 1 + \frac{5/2}{x-3} + \frac{-1/2}{x+1}\,dx$

$= \boxed{x + \frac{5}{2}\ln|x-3| - \frac{1}{2}\ln|x+1| + C}$

Partial Fractions:

$\frac{2x+4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$2x+4 = (x+1)A + (x-3)B$

$x=3$: $10 = 4A \Rightarrow A = 5/2$

$x=-1$: $2 = -4B \Rightarrow B = -1/2$.

(b) $\int y^2(\ln y)^2\,dy$

1) Integration By Parts: $u = (\ln y)^2$, $du = 2\ln y\,\frac{1}{y}$; $dv = y^2\,dy$, $v = \frac{1}{3}y^3$

$= \frac{1}{3}y^3(\ln y)^2 - \frac{2}{3}\int y^2 \ln y\,dy$

2) Integration By Parts: $u = \ln y$, $du = \frac{1}{y}$; $dv = y^2$, $v = \frac{1}{3}y^3$

$= \frac{1}{3}y^3(\ln y)^2 - \frac{2}{3}\left[\frac{1}{3}y^3\ln y - \frac{1}{3}\int y^2\,dy\right]$

$= \frac{1}{3}y^3(\ln y)^2 - \frac{2}{9}y^3\ln y + \frac{2}{9}\int y^2\,dy$

$= \boxed{\frac{1}{3}y^3(\ln y)^2 - \frac{2}{9}y^3\ln y + \frac{2}{27}y^3 + C}$

$= \frac{1}{3}y^3\left[(\ln y)^2 - \frac{2}{3}\ln y + \frac{2}{9}\right] + C$

2. (14 points) Evaluate the following integrals. Simplify but leave your answers in exact form.

(a) $\int_0^4 e^{\sqrt{2t+1}}\,dt$

1) Substitution: $u = \sqrt{2t+1} \Rightarrow u^2 = 2t+1 \Rightarrow \cancel{2}u\,du = \cancel{2}dt$

[or, in 2 steps: $u = 2t+1$, $\frac{1}{2}du = dt$, then $v = \sqrt{u}$, $v^2 = u$, $du = 2v\,dv$]

$= \int_1^3 e^u\, u\,du$

2) I.B.P.: $w = u$, $dw = du$; $dv = e^u du$, $v = e^u$

$= \left[ue^u - e^u\right]\Big|_1^3$

$= (3e^3 - e^3) - (1 \cdot e - e)$

$= \boxed{2e^3}$

(b) $\int_0^{\pi} \sin^3\theta \cos^2\theta\, d\theta$

$= \int_0^{\pi} \sin^2\theta \cos^2\theta \sin\theta\, d\theta$

$= \int_0^{\pi} (1-\cos^2\theta)\cos^2\theta \sin\theta\, d\theta$

$u = \cos\theta$, $du = -\sin\theta\, d\theta$

$= \int_1^{-1} (1-u^2)(u^2)(-1)\,du$

$= \int_{-1}^{1} (u^2 - u^4)\,du = 2\left(\int_0^1 u^2 - u^4\,du\right)$

$= 2\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)\Big|_0^1$

$= 2\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \boxed{\frac{4}{15}}$

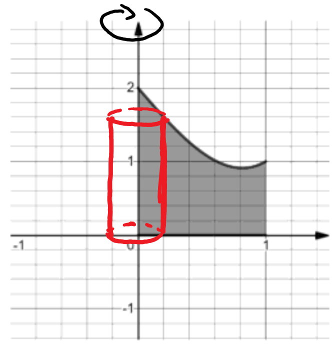
3. (10 points) Consider the area bounded by the curves:

$$y = x^3 - 2x + 2, \qquad x = 0, \qquad x = 1, \qquad \text{and } y = 0.$$

(a) Set up (<u>do not evaluate</u>) an integral equal to the volume swept out by rotating this area about the y-axis.

SHELLS:

$$\boxed{V_1 = \int_0^1 2\pi \underbrace{(x)}_{\text{radius}} \underbrace{(x^3 - 2x + 2)}_{\text{height}}\, dx}$$



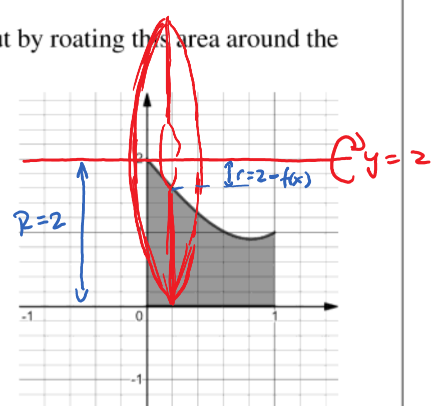
Note: Since we cannot solve $y = x^3 - 2x + 2$ for x in terms of y, we must use vertical rectangles and integrate in x, in all parts of this question.

(b) Set up (<u>do not evaluate</u>) an integral equal to the volume swept out by roating this area around the horizontal line $y = 2$.

WASHERS:

$$\boxed{V_2 = \int_0^1 \pi \underbrace{(2)^2}_{R=2} - \pi \underbrace{(2 - (x^3 - 2x + 2))^2}_{r = 2 - (x^3 - 2x + 2)}\, dx}$$

$$= \pi \int_0^1 4 - (-x^3 + 2x)^2\, dx$$



4. (10 points) A particle moves along a line, with acceleration at t seconds given by:

$$a(t) = \frac{1}{2\sqrt{t}}$$

where the acceleration is measured in $\mathrm{m/sec^2}$. The particle has velocity $v(0) = -2$ m/sec at time $t = 0$ seconds. Find the total distance traveled by the particle during the time interval $0 \le t \le 9$ seconds.

The velocity of the particle at t seconds is:

$$\begin{cases} v(t) = \int \frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\frac{t^{1/2}}{(1/2)} + c = \sqrt{t} + C \\ v(0) = -2 \Rightarrow \quad -2 = \sqrt{0} + c \Rightarrow C = -2 \end{cases}$$

Hence $v(t) = \sqrt{t} - 2$ m/sec

The particle changes direction when $\sqrt{t} - 2 = 0$

$\sqrt{t} = 2$

i.e. at $t = 4$ sec

Note that $v(t) < 0$ for $t < 4$ and $v(t) = \sqrt{t} - 2 > 0$ for $t > 4$.

Total distance for $0 \le t \le 9$ is:

$$\int_0^9 |v(t)|\,dt = \int_0^4 (2 - \sqrt{t})\,dt + \int_4^9 (\sqrt{t} - 2)\,dt$$

$$= \left(2t - \frac{2}{3}t^{3/2}\right)\Big|_0^4 + \left(\frac{2}{3}t^{3/2} - 2t\right)\Big|_4^9$$

$4^{3/2} = (\sqrt{4})^3 = 8$

$$= \left(8 - \frac{16}{3}\right) + \left((18 - 18) - \left(\frac{16}{3} - 8\right)\right)$$

$$= \frac{8}{3} + \frac{8}{3}$$

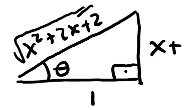
$$= \boxed{16/3 \text{ meters}}$$

5. (10 points) Determine whether the integral $\displaystyle\int_0^\infty \frac{1}{(x^2+2x+2)^{3/2}}\,dx$ is convergent or divergent. If it converges, evaluate it.

Trig sub:

$x+1 = \tan\theta$

$dx = \sec^2\theta\, d\theta$



$$\int \frac{1}{(x^2+2x+2)^{3/2}}\,dx = \int \frac{1}{[(x+1)^2+1]^{3/2}}\,dx$$

$$= \int \frac{1}{(\tan^2\theta+1)^{3/2}} \sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sec^3\theta}\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sec\theta}\, d\theta = \int \cos\theta\, d\theta$$

$$= \sin\theta + C = \frac{x+1}{\sqrt{x^2+2x+2}} + C$$

$$\int_0^\infty \frac{1}{(x^2+2x+2)^{3/2}}\,dx = \lim_{t\to\infty} \left[\frac{x+1}{\sqrt{x^2+2x+2}}\right]\Bigg|_0^t$$

$$= \lim_{t\to\infty}\left(\frac{t+1}{\sqrt{t^2+2t+2}}\right) - \frac{0+1}{\sqrt{0+2(0)+2}}$$

$$= \lim_{t\to\infty}\left(\frac{1+1/t}{\sqrt{1+2/t+2/t^2}}\right) - \frac{1}{\sqrt{2}}$$

$$= \frac{1+0}{\sqrt{1+0+0}} - \frac{1}{\sqrt{2}}$$

$$= 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

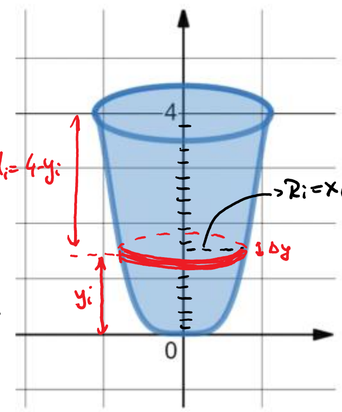
Integral $\boxed{\text{converges to } 1-\frac{1}{\sqrt{2}}}$

6. The portion of the graph of

$$x = \left(\arcsin\left(\frac{y}{4}\right)\right)^{(1/4)} \qquad \text{between } y = 0 \text{ and } y = 4 \text{ meters}$$

is rotated around the y-axis to form a container with depth of 4 meters, as shown. The container is filled with water. The density of water is 1000 kg/m^3 and the gravitational acceleration is 9.8 m/sec^2.

(a) (8 points) Find an integral that gives the work required to pump all the water out over the side. **Do <u>not</u> attempt to evaluate the integral – just set it up.**



Divide $0 \le y \le 4$ into n subintervals

The ith slice of water is a disk of radius $x_i = \left(\arcsin \frac{y_i}{4}\right)^{1/4}$

and thickness Δy, so volume $V_i = \pi R_i^2 \Delta y$

$= \pi \left[\left(\arcsin \frac{y_i}{4}\right)^{1/4}\right]^2 \Delta y$

and weight: $F_i = 9800 V_i = 9800\pi \sqrt{\arcsin\left(\frac{y_i}{4}\right)}\,\Delta y$ Newtons

It must be lifted $d_i = 4 - y_i$ meters.

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F_i d_i = \lim_{n\to\infty} \sum_{i=1}^{n} 9800\pi \sqrt{\arcsin \frac{y_i}{4}}\,(4-y_i)\,\Delta y$$

$$\therefore \boxed{W = \int_0^4 9800\pi (4-y)\sqrt{\arcsin(y/4)}\,dy.}$$

(b) (4 points) Estimate the value of the integral from part (a) using Simpson's Rule with $n = 4$ subintervals. Show work and round your answer to the nearest Joule.

$$\Delta y = \frac{4-0}{4} = 1$$

$$W \cong S_4 = \frac{\Delta y}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right]$$

$$= \frac{1}{3}\, 9800\pi \Big[(4-0)\underbrace{\cancel{\sqrt{\arcsin(0)}}}_{0} + 4(4-1)\sqrt{\arcsin(1/4)} +$$

$$+ 2(4-2)\sqrt{\arcsin(\tfrac{1}{2})} + 4(4-3)\sqrt{\arcsin(3/4)} + \underbrace{\cancel{(4-4)}}_{0}\sqrt{\arcsin(1)}\Big]$$

$$= \frac{9800\pi}{3}\left[12\sqrt{\arcsin(1/4)} + 4\sqrt{\pi/6} + 4\sqrt{\arcsin(3/4)}\right]$$

$$\cong 129{,}411.5414 \text{ Joules} \cong \boxed{129{,}412 \text{ J}}$$

7. (10 points) A hot cube of metal is removed from an oven and left to stand in a cold room. The cube is losing heat in such a way that the rate of change of its temperature is proportional to the difference between the temperature of the metal and the temperature of the room.

Suppose that the room has a constant temperature of 47° F. Initially the cube's temperature is 443° F. Ten minutes after being removed from the oven, the cube's temperature is 380° F.

What will the cube's temperature be 20 minutes after being removed from the oven? Round your answer to two decimal digits.

Let $T(t)$ denote the temperature (in °F) of the cube, t minutes after it's removed from the oven.

It changes according to: $\frac{dT}{dt} = k(T-47)$

so: $\int \frac{1}{T-47}\,dT = \int k\,dt$

$\ln|T-47| = kt + C$

$|T-47| = e^{kt}\cdot e^{C} = Ae^{kt}$

$T = 47 + Ae^{kt}$ for some constants A, k.

Using: $T(0) = 443°F \Rightarrow 443 = 47 + A \Rightarrow A = 443 - 47 = 396$

$\therefore T = 47 + 396e^{kt}$

Also: $T(10) = 380°F \Rightarrow 380 = 47 + 396e^{k(10)} \Rightarrow 396e^{10k} = 333$

$e^{10k} = \frac{333}{396} = \frac{37}{44}$

$k = \frac{1}{10}\ln\left(\frac{333}{396}\right)$

$(\cong -0.017327)$

When $t = 20$ minutes:

$T = 47 + 396e^{\frac{1}{10}\ln(333/396)\cdot 20}$

$= 47 + 396\left(\frac{333}{396}\right)^2$

$= 327.02\overline{27}$

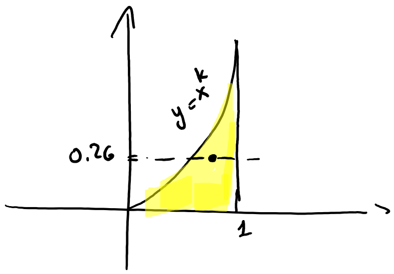
$\cong \boxed{327.02°F}$

8. (10 points) Consider the region in the first quadrant bounded by:

$$y = x^k, \qquad x = 1, \qquad \text{and the } x\text{-axis,}$$

where $k > 0$ is some fixed positive number.

If the center of mass of this region lies on the line $y = 0.26$, what is the value of k?



If $\bar{y} = 0.26$ then $k = ?$

$$A = \int_0^1 x^k\,dx = \frac{x^{k+1}}{k+1}\Big|_0^1 = \frac{1}{k+1}$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(x^k)^2\,dx = \frac{1}{\left(\frac{1}{k+1}\right)}\frac{1}{2}\int_0^1 x^{2k}\,dx = \frac{k+1}{2}\cdot\frac{x^{2k+1}}{2k+1}\Big|_0^1 = \frac{k+1}{4k+2}$$

$$\bar{y} = 0.26 \Rightarrow \frac{k+1}{4k+2} = 0.26$$

$$k+1 = (0.26)(4k+2)$$

$$k+1 = 1.04k + 0.52$$

$$0.04k = 0.48$$

$$k = \frac{0.48}{0.04} = \frac{48}{4}$$

$$\boxed{k = 12}$$

9. (10 points) Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{\ln x}{xy}$$

that satisfies the initial condition $y(1) = 2$. Give your solution in explicit form, $y = f(x)$.

Separating the variables and integrating:

$$\int y\,dy = \int \frac{\ln x}{x}\,dx \qquad u = \ln x,\quad du = \frac{1}{x}dx$$

$$\frac{1}{2}y^2 = \int u\,du$$

$$\frac{1}{2}y^2 = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

$$y^2 = (\ln x)^2 + C_1$$

$$y = \pm\sqrt{(\ln x)^2 + C_1}$$

Since $y(1) = 2$: $\quad 2 = + \sqrt{0^2 + C_1}$

so: $\quad C_1 = 4$ (and we need the positive solution)

Answer: $\boxed{y = \sqrt{(\ln x)^2 + 4}}$